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On Existence of Solutions of *q*-Perturbed Quadratic Integral Equations

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Abstract

We investigate a q-fractional integral equation with supremum and prove an existence theorem for it. We will prove that our q-integral equation has a solution in $C[0,1]$ which is monotonic on $[0,1]$. The monotonicity measures of noncompactness due to Banaś and Olszowy and Darbo's theorem are the main tools used in the proof of our main result.

Keywords

q-Fractional, Integral Equation, Monotonic Solutions, Darbo Theorem, Monotonicity Measure of Noncompactness

1. Introduction

Jackson in [1] introduced the concept of quantum calculus (q-calculus). This area of research has rich history and several applications, see [2]-[4] and references therein. There are several developments and applications of the q-calculus in mathematical physics, especially concerning quantum mechanics, the theory of relativity and special functions [5] [4]. Recently, several researchers attracted their attention by the concept of q-calculus, and we could find several new results in [6] [7] and the references therein.

In several papers among them [8]-[11], integral equations with nonsingular kernels have been studied. In [12]-[14] Darwish *et al*. introduced and studied the quadratic Volterra equations with supremum. Also, Banaś *et al*. and Darwish [13] [15]-[17] studied quadratic integral equations of arbitrary orders with singular kernels. In [18], Darwish generalized and extended Banaś *et al*. [15] results to the perturbed quadratic integral equations of arbitrary orders with singular kernels.

In this paper, we will study the q-perturbed quadratic integral equation with supremum

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$$y(t)=f(t,y(t))+\frac{(\mathcal{A}y)(t)}{\Gamma_q(\beta)}\int_0^t k(t,s)(t-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_q s,\ t\in I=[0,1], \tag{1}$$

where $0<\beta, q\in(0,1), f: I\times\mathbb{R}\to\mathbb{R}$, $\mathcal{A},\mathcal{B}: C(I)\to C(I)$, and $k: I\times I\to\mathbb{R}$.

By using Darbo fixed point theorem and the monotonicity measure of noncompactness due to Banaś and Olszowy [19], we prove the existence of monotonic solution to Equation (1) in $C[0,1]$.

2. q-Calculus and Measure of Noncompactness

First, we collect basic definitions and results of the q-fractional integrals and q-derivatives, for more details, see [5] [6] [20] [21] and references therein.

First, for a real parameter $q\in(0,1)$, we define a q-real number $[a]_q$ by

$$[a]_q=\frac{1-q^a}{1-q},\ a\in\mathbb{R}, \tag{2}$$

and a q-analog of the Pochhammer symbol (q-shifted factorial) is defined by

$$(a;q)_n=\begin{cases}1, & n=0,\\ \prod_{k=1}^{n-1}(1-aq^k), & n\in\mathbb{N}.\end{cases} \tag{3}$$

Also, the q-analog of the power $(a-b)^n$ is given by

$$(a-b)^n=\begin{cases}1, & n=0,\\ \prod_{k=1}^{n-1}(a-bq^k), & n\in\mathbb{N}; a,b\in\mathbb{R}.\end{cases} \tag{4}$$

Moreover,

$$(a-b)^{(n)}=a^n(b/a;q)_n,\ a\neq 0. \tag{5}$$

Notice that, $\lim_{n\to\infty}(a;q)_n$ exists and we will denote it by $(a;q)_\infty$.

More generally, for $\beta\in\mathbb{R}, aq^\beta\neq q^{-n}\ (n\in\mathbb{N})$, we define

$$(a;q)_\beta=\frac{(a;q)_\infty}{(aq^\beta;q)_\infty} \tag{6}$$

and

$$(a-b)^{(\beta)}=a^\beta\frac{(b/a;q)_\infty}{(q^\beta b/a;q)_\infty} \tag{7}$$

Notice that $(a-b)^{(\beta)}=a^\beta(b/a;q)_\beta$. Therefore, if $b=0$, then $a^{(\beta)}=a^\beta$.

The q-gamma function is defined by

$$\Gamma_q(x)=\frac{G(q^x)}{(1-q)^{x-1}G(q)},\ x\in\mathbb{R}\setminus\{0,-1,-2,\cdots\}, \tag{8}$$

where $G(q^x)=\frac{1}{(q^x;q)_\infty}$. Or, equivalently, $\Gamma_q(x)=\frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}$ and satisfies $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$.

Next, the q-derivative of a function f is given by

$$(D_q f)(t)=\frac{f(t)-f(qt)}{t-qt},\ (D_q f)(0)=\lim_{t\to 0}(D_q f)(t), \tag{9}$$

and the q-derivative of higher order of a function f is defined by

$$\left(D_q f\right)(t)=\begin{cases} f(t), & n=0, \\ D_q\left(D_q^{n-1} f\right)(t), & n\in\mathbb{N}. \end{cases} \tag{10}$$

The q-integral of a function f defined on the interval $[0,b]$ is defined by

$$\left(I_q f\right)(t)=\int_a^t f(s)\,\mathrm{d}_q s=t(1-q)\sum_{n=0}^{\infty} q^n f\left(tq^n\right),\ t\in[0,b]. \tag{11}$$

If f is given on the interval $[0,b]$ and $a\in[0,b]$ then

$$\int_a^b f(s)\,\mathrm{d}_q s=\int_0^b f(s)\,\mathrm{d}_q s-\int_0^a f(s)\,\mathrm{d}_q s. \tag{12}$$

The operator I_q^n is defined by

$$\left(I_q^n f\right)(t)=\begin{cases} f(t), & n=0. \\ I_q\left(I_q^{n-1} f\right)(t), & n\in\mathbb{N}, \end{cases} \tag{13}$$

The fundamental theorem of calculus satisfies for D_q and I_q, *i.e.*, $\left(D_q I_q f\right)(t)=f(t)$, and if f is continuous at $t=0$, then $\left(I_q D_q f\right)(t)=f(t)-f(0)$.

The following four formulas will be used later in this paper

$$\begin{aligned} &\left[a(t-s)\right]^{(\beta)}=a^{\beta}(t-s)^{(\beta)} \\ &{}_tD_q(t-s)^{(\beta)}=[\beta]_q(t-s)^{(\beta-1)} \\ &{}_sD_q(t-s)^{(\beta)}=-[\beta]_q(t-qs)^{(\beta-1)} \end{aligned} \tag{14}$$

and

$${}_tD_q\int_0^t f(t,s)\,\mathrm{d}_q s=\int_0^t {}_tD_q f(t,s)\,\mathrm{d}_q s+f(qt,t), \tag{15}$$

where ${}_tD_q$ denotes the q-derivative with respect to variable t.

Notice that, if $\beta>0$ and $a\le b\le t$, then $(t-b)^{(\beta)}\le(t-a)^{(\beta)}$.

Definition 1. [2] Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\beta\ge 0$ is given by

$$\left(I_q^{\beta} f\right)(t)=\begin{cases} f(t), & \beta=0, \\ \dfrac{1}{\Gamma_q(\beta)}\displaystyle\int_0^t (t-qs)^{(\beta-1)} f(s)\,\mathrm{d}_q s=t^{\beta}(1-q)^{\beta}\sum_{n=0}^{\infty} q^n\frac{\left(q^{\beta};q\right)_n}{(q;q)_n} f\left(tq^n\right), & \beta>0, t\in[0,1]. \end{cases} \tag{16}$$

Notice that, for $\beta=1$, the above q-integral reduces to (11).

Definition 2. [2] The fractional q-derivative of the Riemann-Liouville type of order $\beta\ge 0$ is given by

$$\left(D_q^{\beta} f\right)(t)=\begin{cases} f(t), & \beta=0, \\ \left(D_q^{[\beta]} I_q^{[\beta]-\beta} f\right)(t), & \beta>0, \end{cases} \tag{17}$$

where $[\beta]$ denotes the smallest integer greater than or equal to β.

In q-calculus, the derivative rule for the product of two functions and integration by parts formulas are

$$\begin{aligned} &\left(D_q fg\right)(t)=\left(D_q f\right)(t)g(t)+f(qt)\left(D_q g\right)(t), \\ &\int_0^t f(s)\left(D_q g\right)(s)\,\mathrm{d}_q s=\left[f(s)g(s)\right]_0^t-\int_0^t\left(D_q f\right)(s)g(qs)\,\mathrm{d}_q s. \end{aligned} \tag{18}$$

Lemma 1. Let $\gamma,\beta\ge 0$ and f be a function defined on $[0,1]$. Then the following formulas are verified:

$$\begin{aligned} &1)\ \left(I_q^{\gamma} I_q^{\beta} f\right)(t)=\left(I_q^{\gamma+\beta} f\right)(t), \\ &2)\ \left(D_q^{\beta} I_q^{\beta} f\right)(t)=f(t). \end{aligned} \tag{19}$$

Lemma 2. [21] For $\beta > 0$, using q-integration by parts, we have

$$\left(I_q^{\beta}1\right)(t) = \frac{t^{(\beta)}}{\Gamma_q(\beta+1)} \tag{20}$$

or

$$\int_0^t (t-qs)^{(\beta-1)} \mathrm{d}_q s = \frac{t^{(\beta)}}{[\beta]_q}. \tag{21}$$

Second, we recall the basic concepts which we need throughout the paper about measure of noncompactness.

We assume that $(E, \|\cdot\|)$ is a real Banach space with zero element θ and we denote by $B(x,r)$ the closed ball with radius r and centered x, where $B_r \equiv B(\theta, r)$.

Now, let $X \subset E$ and denote by $\bar{X}$ and Conv X the closure and convex closure of X, respectively. Also, the symbols $X+Y$ and λX stands for the usual algebraic operators on sets.

Moreover, the families $\mathfrak{M}_E$ and $\mathfrak{N}_E$ are defined by $\mathfrak{M}_E = \{A \subset E : A \neq \varnothing, A \text{ is bounded}\}$ and $\mathfrak{N}_E = \{B \subset \mathfrak{M}_E : B \text{ is relatively compact}\}$, respectively.

Definition 3. [22] Let $\mu : \mathfrak{M}_E \to \mathbb{R}_+$. If the following conditions

1) $\varnothing \neq \{X \in \mathfrak{M}_E : \mu(X) = 0\} = \ker\mu \subset \mathfrak{N}_E$.

2) $X \subset Y$, then $\mu(X) \le \mu(Y)$.

3) $\mu(X) = \mu(\bar{X}) = \mu(\mathrm{Conv}X)$.

4) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y), 0 \le \lambda \le 1$ and

5) if (X_n) is a sequence of closed subsets of $\mathfrak{M}_E$ with $X_{n+1} \subset X_n, n = 1,2,3,\cdots,$ and $\lim_{n\to\infty}(X_n) = 0$ then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \varnothing$ hold. Then, the mapping μ is said to be a measure of noncompactness in E.

Here, $\ker\mu$ is the kernel of the measure of noncompactness μ.

Our result will establish in $C(I)$ the Banach space of all defined, continuous and real functions on $I \equiv [0,1]$ with $\|y\| = \max_{t\in I}|y(t)|$.

Next, we defined the measure of noncompactness related to monotonicity in $C(I)$, see [19] [22].

We fix a bounded subset $Y \neq \varnothing$ of $C(I)$. For $\varepsilon \ge 0$ and $y \in Y, \omega(y,\varepsilon)$ denotes the modulus of continuity of the function y given by

$$\omega(y,\varepsilon) = \sup\{|y(t) - y(s)| : t,s \in I, |t-s| \le \varepsilon\}. \tag{22}$$

Moreover, we let

$$\omega(Y,\varepsilon) = \sup\{\omega(y,\varepsilon) : y \in Y\} \tag{23}$$

and

$$\omega_0(Y) = \lim_{\varepsilon\to 0}\omega(Y,\varepsilon). \tag{24}$$

Define

$$d(y) = \sup_{s,t\in I, s\le t}\left(|y(t)-y(s)| - [y(t)-y(s)]\right) \tag{25}$$

and

$$d(Y) = \sup_{y\in Y} d(y). \tag{26}$$

Notice that, all functions in Y are nondecreasing on I if and only if $d(Y) = 0$.

Now, we define the map μ on $\mathfrak{M}_{C(I)}$ as

$$\mu(Y) = d(Y) + \omega_0(Y). \tag{27}$$

Clearly, μ verifies all conditions in Definition 3 and, therefore it is a measure of noncompactness in $C(I)$ [19].

Definition 4. Let $\varnothing \neq \Omega \subset E$. Let $\mathcal{P}:\Omega \to E$ be a continuous operator. Suppose that $\mathcal{P}$ maps bounded sets onto bounded ones. If there exists a bounded $Y \subset \Omega$ with $\mu(\mathcal{P}Y) \leq \gamma\mu(Y), \gamma \geq 0$, then $\mathcal{P}$ is said to be satisfies the Darbo condition with respect to a measure of noncompactness μ.

If $\gamma < 1$, then $\mathcal{P}$ is called a contraction operator with respect to μ.

Theorem 1. [23] Let $Q \neq \varnothing$ be a bounded, convex and closed subset of *E*. If $\mathcal{P}:Q \to Q$ is a Contraction operator with respect to μ. Then $\mathcal{P}$ has at least one fixed point belongs to *Q*.

3. Existence Theorem

Let us consider the following suggestions:

a_1) $f: I \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$\exists 0 \leq c < 1 \; s.t. \left| f(t,y) - f(t,x) \right| \leq c\left| y - x \right| \; \forall t \in I \text{ and } x, y \in \mathbb{R}$$

Moreover, $f: I \times \mathbb{R}_+ \to \mathbb{R}$ and $f^* = \max_{t\in I} f(t,0)$.

a_2) The superposition operator *F* generated by the function *f* satisfies for any nonnegative function *y* the condition $d(Fy) \leq cd(y)$, where c is the same constant as in a_1).

a_3) $\mathcal{A}: C(I) \to C(I)$ is a continuous operator which satisfies the Darbo condition for the measure of noncompactness μ with a constant η. Also, $\mathcal{A}y \geq 0$ if $y \geq 0$.

a_4) $\exists a, b \geq 0, s.t. \left|(\mathcal{A}y)(t)\right| \leq a + \|y\| \; \forall y \in C(I), t \in I$.

a_5) The function $k: I \times I \to \mathbb{R}_+$ is continuous on $I \times I$ and nondecreasing $\forall t$ and *s* separately. Moreover, $k^* = \sup_{(t,s)\in I\times I} k(t,s)$.

a_6) $\mathcal{B}: C(I) \to C(I)$ is a continuous operator and there is a nondecreasing function $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|\mathcal{B}y\| \leq \phi(\|y\|)$ for any $y \in C(I)$. Moreover, for every function $y \in C(I)$ which is nonnegative on *I*, the function $\mathcal{B}y$ is nonnegative and nondecreasing on *I*.

a_7) $\exists r_0 > 0$ such that

$$f^* + cr + \frac{(a+br)k^*\phi(r)}{\Gamma_q(\beta+1)} \leq r \tag{28}$$

and $c + \dfrac{\eta k^* \phi(r_0)}{\Gamma_q(\beta+1)} < 1$.

Before, we state and prove our main theorem, we define the two operators $\mathcal{K}$ and $\mathcal{T}$ on $C(I)$ as follows

$$(\mathcal{K}y)(t) = \frac{1}{\Gamma_q(\beta)} \int_0^t k(t,s)(t-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}s \tag{29}$$

and

$$(\mathcal{T}y)(t) = f(t, y(t)) + (\mathcal{A}y)(t)(\mathcal{K}y)(t) \tag{30}$$

respectively. Finding a fixed point of the operator $\mathcal{T}$ defined on the space $C(I)$ is equivalent to solving Equation (1).

Theorem 2. Assume the suggestions (a_1)-(a_7) be verified, then Equation (1) has at least one solution $y \in C(I)$ which is nondecreasing on *I*.

Proof. We divide the proof into seven steps for better readability.

Step 1: We will show that the operator $\mathcal{T}$ maps $C(I)$ into itself.

For this, it is sufficient to show that $\mathcal{K}y \in C(I)$ if $y \in C(I)$. Fix $\varepsilon > 0$ and let $y \in C(I)$ and $t_1, t_2 \in I (t_1 \leq t_2)$ with $|t_2 - t_1| \leq \varepsilon$. We have

$$
\begin{aligned}
&\left|(\mathcal{K}y)(t_2)-(\mathcal{K}y)(t_1)\right| \\
&=\frac{1}{\Gamma_q(\beta)}\left|\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_qs-\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_qs\right| \\
&\le\frac{1}{\Gamma_q(\beta)}\left|\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_qs-\int_0^{t_2}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_qs\right| \\
&\quad+\frac{1}{\Gamma_q(\beta)}\left|\int_0^{t_2}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_qs-\int_0^{t_1}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_qs\right| \\
&\quad+\frac{1}{\Gamma_q(\beta)}\left|\int_0^{t_1}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_qs-\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_qs\right| \\
&\le\frac{1}{\Gamma_q(\beta)}\int_0^{t_2}\left|k(t_2,s)-k(t_1,s)\right|(t_2-qs)^{(\beta-1)}\left|(\mathcal{B}y)(s)\right|\mathrm{d}_qs \\
&\quad+\frac{1}{\Gamma_q(\beta)}\int_{t_1}^{t_2}\left|k(t_1,s)\right|(t_2-qs)^{(\beta-1)}\left|(\mathcal{B}y)(s)\right|\mathrm{d}_qs \\
&\quad+\frac{1}{\Gamma_q(\beta)}\int_0^{t_1}\left|k(t_1,s)\right|\left[(t_1-qs)^{(\beta-1)}-(t_2-qs)^{(\beta-1)}\right]\left|(\mathcal{B}y)(s)\right|\mathrm{d}_qs \\
&\le\frac{\phi(\|y\|)\,\omega_k(\varepsilon,.)}{\Gamma_q(\beta)}\int_0^{t_2}(t_2-qs)^{(\beta-1)}\,\mathrm{d}_qs \\
&\quad+\frac{k^*\phi(\|y\|)}{\Gamma_q(\beta)}\left\{\int_0^{t_1}\left[(t_1-qs)^{(\beta-1)}-(t_2-qs)^{(\beta-1)}\right]\mathrm{d}_qs+\int_{t_1}^{t_2}(t_2-qs)^{(\beta-1)}\,\mathrm{d}_qs\right\} \\
&=\frac{\phi(\|y\|)\,\omega_k(\varepsilon,.)}{\Gamma_q(\beta+1)}t_2^{(\beta)}+\frac{k^*\phi(\|y\|)}{\Gamma_q(\beta+1)}\left[t_1^{(\beta)}-t_2^{(\beta)}+2(t_2-t_1)^{(\beta)}\right] \\
&\le\frac{\phi(\|y\|)\,\omega_k(\varepsilon,.)}{\Gamma_q(\beta+1)}t_2^{\beta}+\frac{2k^*\phi(\|y\|)}{\Gamma_q(\beta+1)}(t_2-t_1)^{(\beta)} \\
&\le\frac{\phi(\|y\|)\,\omega_k(\varepsilon,.)}{\Gamma_q(\beta+1)}+\frac{2k^*\phi(\|y\|)}{\Gamma_q(\beta+1)}\varepsilon^{\beta}.
\end{aligned}
\tag{31}
$$

Notice that, we have used

$$
\omega_k(\varepsilon,.)=\sup\left\{\left|k(t,s)-k(\tau,s)\right|:t,s,\tau\in I\text{ and }|t-\tau|\le\varepsilon\right\}. \tag{32}
$$

Notice that, since the function k is uniformly continuous on $I\times I$, then when $\varepsilon\to 0$ we have that $\omega_k(\varepsilon,.)\to 0$. Thus $\mathcal{K}y\in C(I)$, and therefore, $\mathcal{T}y\in C(I)$.

Step 2: $\mathcal{T}$ applies B_{r_0} into itself.

Now, $\forall t\in I$, we have

$$
\begin{aligned}
\left|(\mathcal{T}y)(t)\right|&\le\left|f(t,y(t))+\frac{(\mathcal{A}y)(t)}{\Gamma_q(\beta)}\int_0^{t}k(t,s)(t-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_qs\right| \\
&\le\left|f(t,y(t))-f(t,0)\right|+\left|f(t,0)\right|+\frac{\left|(\mathcal{A}y)(t)\right|}{\Gamma_q(\beta)}\int_0^{t}\left|k(t,s)\right|(t-qs)^{(\beta-1)}\left|(\mathcal{B}y)(s)\right|\mathrm{d}_qs \\
&\le c\|y\|+f^*+\frac{(a+b\|y\|)k^*\phi(\|y\|)}{\Gamma_q(\beta)}\int_0^{t}(t-qs)^{(\beta-1)}\,\mathrm{d}_qs \\
&=c\|y\|+f^*+\frac{(a+b\|y\|)k^*\phi(\|y\|)}{\Gamma_q(\beta+1)}.
\end{aligned}
\tag{33}
$$

Hence

$$\|\mathcal{T}y\| \le c\|y\| + f^* + \frac{(a+b\|y\|)k^*\phi(\|y\|)}{\Gamma_q(\beta+1)}. \tag{34}$$

Therefore, if $\|y\| \le r_0$ we get from assumption a_7) the following

$$\|\mathcal{T}y\| \le cr_0 + f^* + \frac{(a+br_0)k^*\phi(r_0)}{\Gamma_q(\beta+1)} \le r_0. \tag{35}$$

Therefore, $\mathcal{T}$ maps B_{r_0} into itself.
We define the subset $B_{r_0}^+$ of B_{r_0} by

$$B_{r_0}^+ = \{y \in B_{r_0} : y(t) \ge 0, \text{for } t \in I\} \tag{36}$$

It is clear that $B_{r_0}^+ \ne \varnothing$ is closed, convex and bounded.
Step 3: $\mathcal{T}$ applies the set $B_{r_0}^+$ into itself.
By this facts and suggestions a_1), a_4) and a_6), we obtain $\mathcal{T}$ transforms $B_{r_0}^+$ into itself.
Step 4: The operator $\mathcal{T}$ is continuous on $B_{r_0}^+$.
To prove this, we fix (y_n) to be a sequence in $B_{r_0}^+$ with $y_n \to y$. We will show that $\mathcal{T}y_n \to \mathcal{T}y$.
Thus, we have $\forall t \in I$,

$$\begin{aligned}
&|(\mathcal{T}y_n)(t)-(\mathcal{T}y)(t)| \le |f(t,y_n(t))-f(t,y(t))| \\
&+\left|\frac{(\mathcal{A}y_n)(t)}{\Gamma_q(\beta)}\int_0^t k(t,s)(t-qs)^{(\beta-1)}(\mathcal{B}y_n)(s)\mathrm{d}_qs-\frac{(\mathcal{A}y)(t)}{\Gamma_q(\beta)}\int_0^t k(t,s)(t-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right| \\
&\le c|y_n(t)-y(t)|+\left|\frac{(\mathcal{A}y_n)(t)}{\Gamma_q(\beta)}\int_0^t k(t,s)(t-qs)^{(\beta-1)}(\mathcal{B}y_n)(s)\mathrm{d}_qs-\frac{(\mathcal{A}y)(t)}{\Gamma_q(\beta)}\int_0^t k(t,s)(t-qs)^{(\beta-1)}(\mathcal{B}y_n)(s)\mathrm{d}_qs\right| \\
&+\left|\frac{(\mathcal{A}y)(t)}{\Gamma_q(\beta)}\int_0^t k(t,s)(t-qs)^{(\beta-1)}(\mathcal{B}y_n)(s)\mathrm{d}_qs-\frac{(\mathcal{A}y)(t)}{\Gamma_q(\beta)}\int_0^t k(t,s)(t-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right| \\
&\le c|y_n(t)-y(t)|+\frac{|(\mathcal{A}y_n)(t)-(\mathcal{A}y)(t)|}{\Gamma_q(\beta)}\int_0^t |k(t,s)|(t-qs)^{(\beta-1)}|(\mathcal{B}y_n)(s)|\mathrm{d}_qs \\
&+\frac{|(\mathcal{A}y)(t)|}{\Gamma_q(\beta)}\int_0^t |k(t,s)|(t-qs)^{(\beta-1)}|(\mathcal{B}y_n)(s)-(\mathcal{B}y)(s)|\mathrm{d}_qs.
\end{aligned} \tag{37}$$

Consequently,

$$\|\mathcal{T}y_n-\mathcal{T}y\| \le c\|y_n-y\|+\frac{k^*\phi(r_0)\|\mathcal{A}y_n-\mathcal{A}y\|}{\Gamma_q(\beta+1)}+\frac{(a+br_0)k^*\|\mathcal{B}y_n-\mathcal{B}y\|}{\Gamma_q(\beta+1)}. \tag{38}$$

As $\mathcal{A}$ and $\mathcal{B}$ are continuous operators, $\exists n_1 \in \mathbb{N}$ such that

$$\|\mathcal{A}y_n-\mathcal{A}y\| \le \frac{\varepsilon\Gamma_q(\beta+1)}{3k^*\phi(r_0)}, \quad \forall n \ge n_1. \tag{39}$$

Also, $\exists n_2 \in \mathbb{N}$ such that

$$\|\mathcal{B}y_n-\mathcal{B}y\| \le \frac{\varepsilon\Gamma_q(\beta+1)}{3k^*(a+br_0)}, \quad \forall n \ge n_2. \tag{40}$$

Furthermore, $\exists n_3 \in \mathbb{N}$ such that

$$\|y_n-y\| \le \frac{\varepsilon}{3c}, \quad \forall n \ge n_3. \tag{41}$$

Now, take $\max\{n_1, n_2, n_3\} \le n$, then (38) gives us that

$$\|\mathcal{T}y_n - \mathcal{T}y\| \le \varepsilon . \tag{42}$$

This shows that $\mathcal{T}$ is continuous in $B_{r_0}^+$.

Step 5: In recognition of $\mathcal{T}$ with respect to the quantity ω_0.

Now, we take $\varnothing \ne Y \subset B_{r_0}^+$. Let us fix an arbitrarily number $\varepsilon > 0$ and choose $y \in Y$ and $t_1, t_2 \in I$ with $|t_2 - t_1| \le \varepsilon$. We will be supposed that $t_1 \le t_2$ because no generality will be loss. Then, by using our suggestions and inequality (31), we get

$$\begin{aligned}
&\left|(\mathcal{T}y)(t_2) - (\mathcal{T}y)(t_1)\right| \\
&\le \left|f(t_2, y(t_2)) - f(t_1, y(t_1))\right| + \left|(\mathcal{A}y)(t_2)(\mathcal{K}y)(t_2) - (\mathcal{A}y)(t_2)(\mathcal{K}y)(t_1)\right| \\
&\quad + \left|(\mathcal{A}y)(t_2)(\mathcal{K}y)(t_1) - (\mathcal{A}y)(t_1)(\mathcal{K}y)(t_1)\right| \\
&\le \left|f(t_2, y(t_2)) - f(t_1, y(t_2))\right| + \left|f(t_1, y(t_2)) - f(t_1, y(t_1))\right| \\
&\quad + \left|(\mathcal{A}y)(t_2)\right|\left|(\mathcal{K}y)(t_2) - (\mathcal{K}y)(t_1)\right| + \left|(\mathcal{A}y)(t_2) - (\mathcal{A}y)(t_1)\right|\left|(\mathcal{K}y)(t_1)\right| \\
&\le \gamma_{r_0}(f, \varepsilon) + c\omega(y, \varepsilon) + \left|(\mathcal{A}y)(t_2)\right|\left|(\mathcal{K}y)(t_2) - (\mathcal{K}y)(t_1)\right| \\
&\quad + \left|(\mathcal{A}y)(t_2) - (\mathcal{A}y)(t_1)\right|\left|(\mathcal{K}y)(t_1)\right| \\
&\le \gamma_{r_0}(f, \varepsilon) + c\omega(y, \varepsilon) + \frac{(a + b\|y\|)\phi(\|y\|)}{\Gamma_q(\beta+1)}\left[\omega_k(\varepsilon, .) + 2k^*\varepsilon^\beta\right] \\
&\quad + \frac{\omega(\mathcal{A}y, \varepsilon)}{\Gamma_q(\beta+1)} k^* \phi(\|y\|) \\
&\le \gamma_{r_0}(f, \varepsilon) + c\omega(y, \varepsilon) + \frac{(a + br_0)\phi(r_0)}{\Gamma_q(\beta+1)}\left[\omega_k(\varepsilon, .) + 2k^*\varepsilon^\beta\right] \\
&\quad + \frac{\omega(\mathcal{A}y, \varepsilon)}{\Gamma_q(\beta+1)} k^* \phi(r_0).
\end{aligned} \tag{43}$$

The last estimate implies

$$\begin{aligned}
\omega(\mathcal{T}y, \varepsilon) &\le \gamma_{r_0}(f, \varepsilon) + c\omega(y, \varepsilon) \\
&\quad + \frac{(a + br_0)\phi(r_0)}{\Gamma_q(\beta+1)}\left[\omega_k(\varepsilon, .) + 2k^*\varepsilon^\beta\right] + \frac{\omega(\mathcal{A}y, \varepsilon)}{\Gamma_q(\beta+1)} k^* \phi(r_0)
\end{aligned} \tag{44}$$

and, consequently,

$$\begin{aligned}
\omega(\mathcal{T}Y, \varepsilon) &\le \gamma_{r_0}(f, \varepsilon) + c\omega(Y, \varepsilon) \\
&\quad + \frac{(a + br_0)\phi(r_0)}{\Gamma_q(\beta+1)}\left[\omega_k(\varepsilon, .) + 2k^*\varepsilon^\beta\right] + \frac{\omega(\mathcal{A}Y, \varepsilon)}{\Gamma_q(\beta+1)} k^* \phi(r_0).
\end{aligned} \tag{45}$$

Since the function *k* is uniformly continuous on $I \times I$ and the function *f* is continuous on $I \times [0, r_0]$, then the last inequality gives us that

$$\omega_0(\mathcal{T}Y) \le c\omega_0(Y) + \frac{k^*\phi(r_0)}{\Gamma_q(\beta+1)}\omega_0(\mathcal{A}Y). \tag{46}$$

Step 6: In recognition of $\mathcal{T}$ with respect to the quantity *d*.

Here, we fix an arbitrary $y \in Y$ and $t_1, t_2 \in I$ with $t_2 > t_1$. Then, by our assumption, we obtain our suggestions, we have

$$
\begin{aligned}
&\left|(\mathcal{T}y)(t_2)-(\mathcal{T}y)(t_1)\right|-\left[(\mathcal{T}y)(t_2)-(\mathcal{T}y)(t_1)\right]\\
&=\left|f(t_2,y(t_2))+\frac{(\mathcal{A}y)(t_2)}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right.\\
&\quad\left.-f(t_1,y(t_1))-\frac{(\mathcal{A}y)(t_1)}{\Gamma_q(\beta)}\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right|\\
&\quad-\left[f(t_2,y(t_2))+\frac{(\mathcal{A}y)(t_2)}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right.\\
&\quad\left.-f(t_1,y(t_1))-\frac{(\mathcal{A}y)(t_1)}{\Gamma_q(\beta)}\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right]\\
&\le\left\{\left|f(t_2,y(t_2))-f(t_1,y(t_1))\right|-\left[f(t_2,y(t_2))-f(t_1,y(t_1))\right]\right\}\\
&\quad+\left|\frac{(\mathcal{A}y)(t_2)}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\frac{(\mathcal{T}x)(t_1)}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right|\\
&\quad+\left|\frac{(\mathcal{A}y)(t_1)}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\frac{(\mathcal{T}x)(t_1)}{\Gamma_q(\beta)}\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right|\\
&\quad-\left\{\left[\frac{(\mathcal{A}y)(t_2)}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\frac{(\mathcal{T}x)(t_1)}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right]\right.\\
&\quad\left.+\left[\frac{(\mathcal{A}y)(t_1)}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\frac{(\mathcal{T}x)(t_1)}{\Gamma_q(\beta)}\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right]\right\}\\
&\le\left\{\left|f(t_2,y(t_2))-f(t_1,y(t_1))\right|-\left[f(t_2,y(t_2))-f(t_1,y(t_1))\right]\right\}\\
&\quad+\left\{\left|(\mathcal{A}y)(t_2)-(\mathcal{A}y)(t_1)\right|-\left[(\mathcal{A}y)(t_2)-(\mathcal{A}y)(t_1)\right]\right\}\frac{1}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\\
&\quad+\frac{(\mathcal{A}y)(t_1)}{\Gamma_q(\beta)}\left\{\left|\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right|\right.\\
&\quad\left.-\left[\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\right]\right\}.
\end{aligned}
\tag{47}
$$

Now, we will prove that

$$
\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\ge 0. \tag{48}
$$

We find that

$$
\begin{aligned}
&\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\\
&=\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\int_0^{t_2}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\\
&\quad+\int_0^{t_2}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\int_0^{t_1}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\\
&\quad+\int_0^{t_1}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs-\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\\
&=\int_0^{t_2}\left(k(t_2,s)-k(t_1,s)\right)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs+\int_{t_1}^{t_2}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\mathrm{d}_qs\\
&\quad+\int_0^{t_1}k(t_1,s)\left[(t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}\right](\mathcal{B}y)(s)\mathrm{d}_qs.
\end{aligned}
\tag{49}
$$

But, $k(t_1,s)\le k(t_2,s)$ because $k(t,s)$ is increasing with respect to t, then

$$\int_0^{t_2}\left(k(t_2,s)-k(t_1,s)\right)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_q s\ge 0, \tag{50}$$

and, since $(t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}$ is negative for $s\in[0,t_1)$ then

$$\begin{aligned}&\int_0^{t_1}k(t_1,s)\left[(t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}\right](\mathcal{B}y)(s)\,\mathrm{d}_q s+\int_{t_1}^{t_2}k(t_1,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_q s\\ &\ge\int_0^{t_1}k(t_1,t_1)\left[(t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}\right](\mathcal{B}y)(t_1)\,\mathrm{d}_q s+\int_{t_1}^{t_2}k(t_1,t_1)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(t_1)\,\mathrm{d}_q s\\ &=k(t_1,t_1)(\mathcal{B}y)(t_1)\left[\int_0^{t_2}(t_2-qs)^{(\beta-1)}\,\mathrm{d}_q s-\int_0^{t_1}(t_1-qs)^{(\beta-1)}\,\mathrm{d}_q s\right]\\ &=k(t_1,t_1)\frac{t_2^{(\beta)}-t_1^{(\beta)}}{[\beta]_q}(\mathcal{B}y)(t_1)\ge 0.\end{aligned} \tag{51}$$

Inequalities (50) and (51) imply that

$$\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_q s-\int_0^{t_1}k(t_1,s)(t_1-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_q s\ge 0.$$

This inequality and (47) gives us

$$\begin{aligned}&\left|(\mathcal{T}y)(t_2)-(\mathcal{T}y)(t_1)\right|-\left[(\mathcal{T}y)(t_2)-(\mathcal{T}y)(t_1)\right]\\ &\le\left\{\left|f(t_2,y(t_2))-f(t_1,y(t_1))\right|-\left[f(t_2,y(t_2))-f(t_1,y(t_1))\right]\right\}\\ &\quad+\left\{\left|(\mathcal{A}y)(t_2)-(\mathcal{A}y)(t_1)\right|-\left[(\mathcal{A}y)(t_2)-(\mathcal{A}y)(t_1)\right]\right\}\\ &\quad\times\frac{1}{\Gamma_q(\beta)}\int_0^{t_2}k(t_2,s)(t_2-qs)^{(\beta-1)}(\mathcal{B}y)(s)\,\mathrm{d}_q s\\ &\le d(Fy)+\frac{k^*\phi(r_0)}{\Gamma_q(\beta+1)}d(\mathcal{A}y).\end{aligned} \tag{52}$$

The above estimate implies that

$$d(\mathcal{T}y)\le cd(y)+\frac{k^*\phi(r_0)}{\Gamma_q(\beta+1)}d(\mathcal{A}y). \tag{53}$$

Therefore,

$$d(\mathcal{T}Y)\le cd(Y)+\frac{k^*\phi(r_0)}{\Gamma_q(\beta+1)}d(\mathcal{A}Y). \tag{54}$$

Step 7: $\mathcal{T}$ is contraction with respect to the measure of noncompactness μ.
Inequalities (46) and (54) give us that

$$\omega_0(\mathcal{T}Y)+d(\mathcal{T}Y)\le c\left(\omega_0(Y)+d(Y)\right)+\frac{k^*\phi(r_0)}{\Gamma_q(\beta+1)}\left(\omega_0(\mathcal{A}Y)+d(\mathcal{A}Y)\right) \tag{55}$$

or

$$\mu(\mathcal{T}Y)\le c\mu(Y)+\frac{k^*\phi(r_0)}{\Gamma_q(\beta+1)}\mu(\mathcal{A}Y)\le\left(c+\frac{\eta k^*\phi(r_0)}{\Gamma_q(\beta+1)}\right)\mu(Y). \tag{56}$$

But $c+\dfrac{\eta k^*\phi(r_0)}{\Gamma_q(\beta+1)}<1$, then

$$\mu(\mathcal{T}Y)\le\mu(Y). \tag{57}$$

Inequality (57) enables us to use Theorem 1, then there are solutions to Equation (1) in $C(I)$. This finishes our proof.

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