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KAUFFMAN BRACKET OF 2- AND 3-STRAND BRAID LINKS

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ABSTRACT. In this paper we give explicit formulas of the Kauffman bracket of the 2-strand braid link $\widehat{x_1^n}$ and the 3-strand braid link $\widehat{x_1^b x_2^m}$. We also show that the Kauffman bracket of the 3-strand braid link $\widehat{x_1^b x_2^m}$ is actually the product of the Kauffman brackets of the 2-strand braid links $\widehat{x_1^b}$ and $\widehat{x_1^m}$.

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Key words and phrases: kauffman bracket; braid link.

1. Introduction

The Kauffman bracket was introduced by L. H. Kauffman in 1987 in [1].
The Kauffman bracket (polynomial) is actually not a knot invariant because it is not invariant under the first Reidemeister move. However, it has many applications and it can be extended to the popular Jones polynomial, which is invariant under all three Reidemeister moves. In the present work we shall confine ourselves to the Kauffman bracket to avoid from unnecessary length and to leave it for applications. In [2] Nizami et al, computed Khavanov Homology of Braid Links and in [3] gave recursive form of Kauffman Bracket.
This paper is organized as follows: In Section 2 we shall give the basic ideas about knots, braids, and the Kauffman bracket. In Section 3 we shall present the main results.

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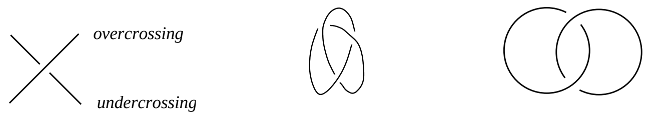
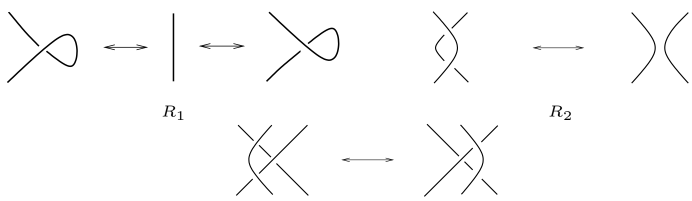
2. Preliminary Notions

2.1. Links. A *link* is a disjoint union of circles embedded in $\mathbb{R}^3$. A one-component link is called a *knot*. Links are usually studied via projecting them on a plan; a projection with extra information of *overcrossing* and *undercrossing* is called the *link diagram*.



Trefoil knot *Hopf link*

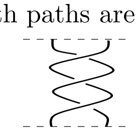
Two links are *isotopic* iff one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself. A fundamental result by Reidemeister [4] about the isotopic link diagrams is: *Two unoriented links* L_1 *and* L_2 *are equivalent if and only if a diagram of* L_1 *can be transformed into a diagram of* L_2 *by a finite sequence of ambient isotopies of the plane and the local (Reidemeister) moves of the following three types:*



R_1 R_2 R_3

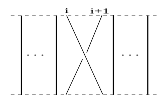
The set of all links that are equivalent to a link L is called a *class* of L. By a link L we shall always mean the class of L.
The main question of knot theory is *which two links are equivalent and which are not*? To address this question one needs a *knot invariant*, a function that gives one value on all links that belong to a single class and gives different values (but not always) on knots that belong to different classes. The present work is basically concerned with this question.

2.2. Braids. Braids were first studied by Emil Artin in 1925 [5, 6], which play an important role in knot theory, see [7, 8] for detail.
An n-strand *braid* is a set of n non intersecting smooth paths connecting n points on a horizontal plane to n points exactly below them on another horizontal plane in an arbitrary order. The smooth paths are called *strands* of the braid.



A 2-strand braid

The *product ab* of two n-strand braids is defined by putting b above a and gluing their end points.
A braid with only one crossing is called *elementary* braid. The ith elementary braid x_i on n strands is:



x_i

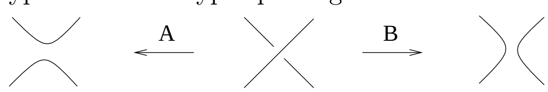
A useful property of elementary braids is that every braid can be written as a product of elementary braids. For instance, the above 2-strand braid is $x_i^{-3} = (x_i^{-1})(x_i^{-1})(x_i^{-1})$.
The *closure* of a braid b is the link $\widehat{b}$ obtained by connecting the lower ends of b with the corresponding upper ends.

b $\widehat{b}$

An important result by Alexander [9] connecting knots and braids is: *Each link can be represented as the closure of a braid.*

Remark 2.1. In the last section we shall present all the links as closures of products of elementary braids.

2.3. The Kauffman Bracket. The Kauffman bracket was introduced by Kauffman in [10].
Before the definition it is better to understand the two types of splitting of a crossing, the A-type and the B-type splittings:



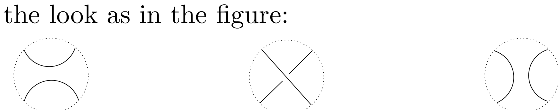
A- and B-type splittings

In the following, the symbols $\bigcirc$ and $\bigsqcup$ represent respectively the unknot and the disconnected sum.

Definition 2.1. The *Kauffman bracket* is the function $\langle\cdot\rangle$: Links $\to \mathbb{Z}[a, a^{-1}]$ defined by the axioms:

$$\begin{aligned} \langle L\rangle &= a\langle L_A\rangle + a^{-1}\langle L_B\rangle \\ \langle L \sqcup \bigcirc\rangle &= (-a^2 - a^{-2})\langle L\rangle \\ \langle\bigcirc\rangle &= 1 \end{aligned}$$

Here L, L_A, and L_B are three links which are isotopic everywhere except at one crossing where the look as in the figure:



L_A L L_B

Proposition 2.2. *The Kauffman polynomial is invariant under second and third Reidemeister moves but not under the first Reidemeister move.*

3. Main Results

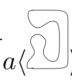
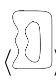
In this section we shall give the Kauffman bracket of the links $\widehat{x_1^n}$ and $\widehat{x_1^b x_2^m}$, and show that $\langle\widehat{x_1^b x_2^m}\rangle = \langle\widehat{x_1^b}\rangle\langle\widehat{x_2^m}\rangle$.
The links $\widehat{x_1^n}$ fall into two categories, the two-component links when n is even and the one-component links (means knots) when n is odd. When n is even, we have:

Proposition 3.1. *The Kauffman bracket of the link $\widehat{x_1^n}$, when $n \geq 2$ is even, is*

$$< \widehat{x_1^n} >= -a^{3n-2} + a^{3n-6} - a^{3n-10} + a^{3n-14} - \cdots - a^{-n+6} - a^{-n-2}. \tag{1}$$

Proof. We prove it by induction on n.
When $n = 2$,

$\langle\widehat{x_1^2}\rangle = \langle$  $\rangle = a\langle$  $\rangle + a^{-1}\langle$  $\rangle$

$= a\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big] + a^{-1}\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big]$

$$\begin{aligned} &= a\big[a(-a^2 - a^{-2}) + a^{-1}(1)\big] + a^{-1}\big[a(1) + a^{-1}(-a^2 - a^{-2})\big] \\ &= -a^4 - a^{-4}. \end{aligned} \tag{2}$$

Similarly, we have

$$\begin{aligned} \langle\widehat{x_1^4}\rangle &= -a^{10} + a^6 - a^2 - a^{-6} \\ &= -a^{3(4)-2} + a^{3(4)-6} + a^{-2}\langle\widehat{x_1^2}\rangle \end{aligned} \tag{3}$$

and

$$\begin{aligned} \langle\widehat{x_1^6}\rangle &= -a^{16} + a^{12} - a^8 + a^4 - a^0 - a^{-8} \\ &= -a^{3(6)-2} + a^{3(6)-6} + a^{-2}\langle\widehat{x_1^4}\rangle. \end{aligned} \tag{4}$$

We now assume the result holds for $n = k$, that is

$$\langle\widehat{x_1^k}\rangle = -a^{3k-2} + a^{3k-6} - a^{3k-10} + a^{3k-14} - \cdots - a^{6-k} - a^{-k-2}. \tag{5}$$

Now for $n = k + 1$, we, following Equations 3.3 and 3.4, write

$$\begin{aligned} \langle\widehat{x_1^{k+2}}\rangle &= -a^{3(k+2)-2} + a^{3(k+2)-6} + a^{-2}\langle\widehat{x_1^k}\rangle \\ &= -a^{3(k+2)-2} + a^{3(k+2)-6} + a^{-2}\Big[- a^{3k-2} + a^{3k-6} - a^{3k-10} \\ &\quad +a^{3k-14} - \cdots - a^{6-k} - a^{-k-2}\Big] \\ &= -a^{3(k+2)-2} + a^{3(k+2)-6} - a^{3k-4} + a^{3k-8} - a^{3k-12} + a^{3k-16} \\ &\quad - \cdots - a^{4-k} - a^{-k-4} \\ &= -a^{3(k+2)-2} + a^{3(k+2)-6} - a^{3(k+2)-10} + a^{3(k+2)-14} - a^{3(k+2)-18} \\ &\quad +a^{3(k+2)-22} - \cdots - a^{6-(k+2)} - a^{-(k+2)-2} \end{aligned}$$

This completes the proof. □

Proposition 3.2. *The Kauffman bracket of the knots $\widehat{x_1^n}$, when n is odd, is*

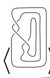
$$< \widehat{x_1^n} >= a^{3n-2} - a^{3n-6} + a^{3n-10} - a^{3n-14} + \cdots - a^{-n+6} - a^{-n-2}. \quad (6)$$

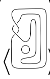
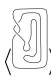
Proof. Similar to the proof of Proposition 3.1 □

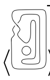
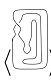
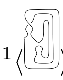
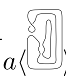
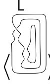
Proposition 3.3. *The Kauffman bracket of the braid link $\widehat{x_1^b x_2^b}$, when b is even, is*

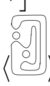
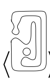
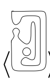
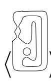
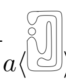
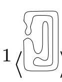
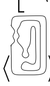
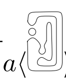
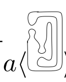
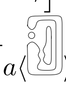
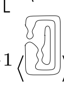
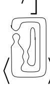
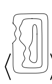
$$\langle \widehat{x_1^b x_2^b} \rangle = \sum_{i=1}^{b-1} i(-1)^{i+1} a^{6b-4i} + \sum_{i=1}^{b} (-1)^{i+1}(b-i)a^{2b-4i} - (b-2)a^{2b} + a^{4-2b} + a^{-2b-4}.$$

Proof. We prove it by induction on b.
When $b = 2$, we have

$\langle \widehat{x_1^2 x_2^2} \rangle = \langle$  $\rangle = a\langle$  $\rangle + a^{-1}\langle$  $\rangle$

$= a\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big] + a^{-1}\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big]$

$= a^2\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big] + a\langle$  $\rangle + a^{-1}\langle$  $\rangle + a\langle$  $\rangle + a^{-1}\langle$  $\rangle + a^{-2}\Big[a\langle$  $\rangle +$
$a^{-1}\langle$  $\rangle\Big]$

$= a^3\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big] + a\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big] + a\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big] + a^{-1}\Big[a\langle$  $\rangle +$
$a^{-1}\langle$  $\rangle\Big] + a\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big] + a^{-1}\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big] + a^{-1}\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big] +$
$a^{-3}\Big[a\langle$  $\rangle + a^{-1}\langle$  $\rangle\Big]$

$= a^4(-a^2 - a^{-2})^2 + a^2(-a^2 - a^{-2}) + a^2(-a^2 - a^{-2}) + (-a^2 - a^{-2})^2 + a^2(-a^2 - a^{-2}) + 1 + 1 + a^{-2}(-a^2 - a^{-2}) + a^2(-a^2 - a^{-2}) + 1 + 1 + a^{-2}(-a^2 - a^{-2}) + (-a^2 - a^{-2})^2 + a^{-2}(-a^2 - a^{-2}) + a^{-2}(-a^2 - a^{-2}) + a^{-4}(-a^2 - a^{-2})^2$
$= a^8 + 2 + a^{-8} = \big[a^8\big] + \big[1\big] + \big[0\big] + \big[1 + a^{-8}\big]$
$= \sum_{i=1}^{1} i(-1)^{i+1} a^{6(2)-4i} + \sum_{i=1}^{2} (-1)^{i+1}(2-i)a^{2(2)-4i} - (2-2)a^{2(2)} + a^{4-2(2)} + a^{-2(2)-4}$,
as required.
Similarly, we get

$$\begin{aligned} \langle \widehat{x_1^4 x_2^4} \rangle &= a^{20} - 2a^{16} + 3a^{12} - 2a^8 + 3a^4 - 2 + 2a^{-4} + a^{-12} \quad (7) \\ &= \big[a^{20} - 2a^{16} + 3a^{12}\big] + \big[3a^4 - 2 + a^{-4}\big] - 2a^8 + \big[a^{-4} + a^{-12}\big] \end{aligned}$$

$$= \sum_{i=1}^{3} i(-1)^{i+1}a^{24-4i} + \sum_{i=1}^{4}(-1)^{i+1}(4-i)a^{8-4i} - 2a^8 + a^{-4} + a^{-12}.$$

In order to manage the proof, we reorganize (3.7):

$$\begin{aligned}\langle \widehat{x_1^4 x_2^4}\rangle &= \big[a^4 + 2a^{-4} + a^{-12}\big] - [a^4] + [a^{20}] - 2 + \big[-2a^{16} + 3a^{12}\big] \\ &\quad + \big[-2a^8 + 3a^4\big] \\ &= a^{-4}\big[\langle \widehat{x_1^2 x_2^2}\rangle\big] - \sum_{i=1}^{1} i(-1)^{i+1}a^{8-4i} + \sum_{i=1}^{1} i(-1)^{i+1}a^{24-4i} - 2 \\ &\quad + \sum_{i=2}^{3} i(-1)^{i+1}a^{24-4i} - 2a^8 + 3a^4 \end{aligned} \tag{8}$$

Similarly,

$$\begin{aligned}\langle \widehat{x_1^6 x_2^6}\rangle &= a^{32} - 2a^{28} + 3a^{24} - 4a^{20} + 5a^{16} - 4a^{12} + 5a^8 - 4a^4 + 3 - 2a^{-4} \\ &\quad + 2a^{-8} + a^{-16} \\ &= a^{-4}\big[\langle \widehat{x_1^4 x_2^4}\rangle\big] - \sum_{i=1}^{3} i(-1)^{i+1}a^{20-4i} + \sum_{i=1}^{3} i(-1)^{i+1}a^{36-4i} - 2a^4 \\ &\quad + \sum_{i=4}^{5} i(-1)^{i+1}a^{36-4i} - 4a^{12} + 5a^8 \end{aligned} \tag{9}$$

Deducting from Equations 3.9 and 3.10, we can write

$$\begin{aligned}\langle \widehat{x_1^b x_2^b}\rangle &= a^{-4}\big[\langle \widehat{x_1^{b-2} x_2^{b-2}}\rangle\big] - \sum_{i=1}^{b-3} i(-1)^{i+1}a^{6b-4i-16} + \sum_{i=1}^{b-3} i(-1)^{i+1}a^{6b-4i} \\ &\quad -2a^{2b-8} + \sum_{i=b-2}^{b-1} i(-1)^{i+1}a^{6b-4i} - (b-2)a^{2b} + (b-1)a^{2b-4}. \end{aligned}$$

We now assume the result holds for $b = k$, that is

$$\begin{aligned}\langle \widehat{x_1^k x_2^k}\rangle &= \sum_{i=1}^{k-1} i(-1)^{i+1}a^{6k-4i} + \sum_{i=1}^{k}(-1)^{i+1}(k-i)a^{2k-4i} - (k-2)a^{2k} \\ &\quad + a^{4-2k} + a^{-2k-4}. \end{aligned} \tag{10}$$

Now for $b = k+2$, we have

$$\begin{aligned}\langle \widehat{x_1^{k+2} x_2^{k+2}}\rangle &= a^{-4}\big[\langle \widehat{x_1^k x_2^k}\rangle\big] - \sum_{i=1}^{k-1} i(-1)^{i+1}a^{6k-4i-4} + \sum_{i=1}^{k-1} i(-1)^{i+1}a^{6k-4i+12} \\ &\quad -2a^{2k-4} + \sum_{i=k}^{k+1} i(-1)^{i+1}a^{6k-4i+12} - ka^{2k+4} + (k+1)a^{2k} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{k-1} i(-1)^{i+1}a^{6k-4i-4} + \sum_{i=1}^{k}(-1)^{i+1}(k-i)a^{2k-4i-4} \\
&\quad -(k-2)a^{2k-4} + a^{-2k} + a^{-2k-8} \\
&\quad -\sum_{i=1}^{k-1} i(-1)^{i+1}a^{6k-4i-4} + \sum_{i=1}^{k-1} i(-1)^{i+1}a^{6k-4i+12} - 2a^{2k-4} \\
&\quad +\sum_{i=k}^{k+1} i(-1)^{i+1}a^{6k-4i+12} - ka^{2k+4} + (k+1)a^{2k} \\
&= \sum_{i=1}^{k+1} i(-1)^{i+1}a^{6k-4i+12} + \sum_{i=-1}^{k}(-1)^{i+1}(k-i)a^{2k-4i-4} \\
&\quad +a^{-2k} + a^{-2k-8} - ka^{2k+4} \\
&= \sum_{i=1}^{k+1} i(-1)^{i+1}a^{6k-4i+12} + \sum_{i=1}^{k+2}(-1)^{i+1}(k+2-i)a^{2k-4i+4} \\
&\quad +a^{-2k} + a^{-2k-8} - ka^{2k+4},
\end{aligned}$$

and the induction is completed. □

Proposition 3.4. *The Kauffman bracket of the braid link $\widehat{x_1^b x_2^b}$, when b is odd, is*

$$\begin{aligned}
\langle \widehat{x_1^b x_2^b} \rangle &= \sum_{i=1}^{b-1} i(-1)^{i+1}a^{6b-4i} + \sum_{i=1}^{b}(-1)^{i}(b-i)a^{2b-4i} + (b-2)a^{2b} \\
&\quad +a^{4-2b} + a^{-2b-4}.
\end{aligned}$$

Proof. Similar to the proof of proposition 3.3. □

Proposition 3.5. *The Kauffman bracket of $\widehat{x_1^b x_2^m}$, when $b > m \geq 2$, is*

$$\begin{aligned}
\langle \widehat{x_1^b x_2^m} \rangle &= \sum_{i=1}^{m-1}(-1)^{b+m+1-i}(i)a^{3(b+m)-4i} + (-1)^{b+1}(m-1)a^{3b-m} \\
&\quad +m\sum_{i=1}^{b-m-1}(-1)^{b+1-i}a^{3b-m-4i} + (-1)^{m+1}(m-1)a^{-b+3m} \\
&\quad +\sum_{i=1}^{m-2}(-1)^{m+1-i}(m-i)a^{-b+3m-4i} + 2a^{-b-m+4} + a^{-b-m-4}.
\end{aligned}$$

Proof. We first verify the result for arbitrary b and $m = 2$:
Resolving all 2^{3+2} crossings as were resolved for $\langle \widehat{x_1^2 x_2^2} \rangle$ in Proposition 3.3, we get

$$\langle \widehat{x_1^3 x_2^2} \rangle = -a^{11} + a^7 - a^3 + 2a^{-1} + a^{-9}$$

Similarly, we get

$$\begin{aligned}\langle\widehat{x_1^4x_2^2}\rangle &= a^{14}-a^{10}+2a^6-a^2+2a^{-2}+a^{-10}\\ &= -a^3\langle\widehat{x_1^3x_2^2}\rangle+a^6+a^2+2a^{-2}+a^{-10}+a^{-6} \quad (11)\\ \langle\widehat{x_1^5x_2^2}\rangle &= -a^{17}+a^{13}-2a^9+2a^5-a+2a^{-3}+a^{-11}\\ &= -a^3\langle\widehat{x_1^4x_2^2}\rangle+a^5+a+2a^{-3}+a^{-11}+a^{-7} \quad (12)\\ \langle\widehat{x_1^6x_2^2}\rangle &= a^{20}-a^{16}+2a^{12}-2a^8+2a^4-1+2a^{-4}+a^{-12}\\ &= -a^3\langle\widehat{x_1^5x_2^2}\rangle+a^4+1+2a^{-4}+a^{-8}+a^{-12} \quad (13)\end{aligned}$$

It follows from (3.11), (3.12), and (3.13) that

$$\langle\widehat{x_1^bx_2^2}\rangle = -a^3\langle\widehat{x_1^{b-1}x_2^2}\rangle+a^{-b+10}+a^{-b+6}+2a^{-b+2}+a^{-b-2}+a^{-b-6}.$$

Now suppose the result is true for $b=t$ and $m=2$, that is

$$\begin{aligned}\langle\widehat{x_1^tx_2^2}\rangle &= (-1)^{-t+2}a^{3t+2}+(-1)^{t+1}a^{3t-2}+2\sum_{i=1}^{t-3}(-1)^{t+1-i}a^{3t-2-4i}\\ &\quad -a^{-t+6}+2a^{-t+2}+a^{-t-6}. \quad (14)\end{aligned}$$

For $b=t+1$, we have

$$\begin{aligned}\langle\widehat{x_1^{t+1}x_2^2}\rangle &= -a^3\langle\widehat{x_1^tx_2^2}\rangle+a^{-t+9}+a^{-t+5}+2a^{-t+1}+a^{-t-3}+a^{-t-7}\\ &= -a^3\Big[(-1)^{-t+2}a^{3t+2}+(-1)^{t+1}a^{3t-2}+2\sum_{i=1}^{t-3}(-1)^{t+1-i}a^{3t-2-4i}\\ &\quad -a^{-t+6}+2a^{-t+2}+a^{-t-6}\Big]+a^{-t+9}+a^{-t+5}+2a^{-t+1}\\ &\quad +a^{-t-3}+a^{-t-7}\\ &= (-1)^{t+3}a^{3t+5}+(-1)^{t+2}a^{3t+1}+2\sum_{i=1}^{t-3}(-1)^{t+2-i}a^{3t+1-4i}\\ &\quad +a^{-t+9}-2a^{-t+5}-a^{-t-3}+a^{-t+9}+a^{-t+5}+2a^{-t+1}\\ &\quad +a^{-t-3}+a^{-t-7}\\ &= (-1)^{t+3}a^{3t+5}+(-1)^{t+2}a^{3t+1}\\ &\quad +\Big[2\sum_{i=1}^{t-3}(-1)^{t+2-i}a^{3t+1-4i}+2a^{-t+9}\Big]\\ &\quad -a^{-t+5}+2a^{-t+1}+a^{-t-7}\\ &= (-1)^{(t+1)+2}a^{3(t+1)+2}+(-1)^{(t+1)+1}a^{3(t+1)-2}\\ &\quad +2\sum_{i=1}^{(t+1)-3}(-1)^{(t+1)+1-i}a^{3(t+1)-2-4i}\\ &\quad -a^{-(t+1)+6}+2a^{-(t+1)+2}+a^{-(t+1)-6}.\end{aligned}$$

Similarly, we get

$$\langle\widehat{x_1^b x_2^3}\rangle = \sum_{i=1}^{2}(-1)^{b+4-i}(i)a^{3b+9-4i} + (-1)^{b+1}2a^{3b-3} + 3\sum_{i=1}^{b-4}(-1)^{b+1-i}a^{3b-3-4i} + 2a^{-b+9} - a^{-b+5} + 2a^{-b+1} + a^{-b-7}$$

and

$$\langle\widehat{x_1^b x_2^4}\rangle = \sum_{i=1}^{3}(-1)^{b+5-i}(i)a^{3b+12-4i} + (-1)^{b+1}3a^{3b-4} + 4\sum_{i=1}^{b-5}(-1)^{b+1-i}a^{3b-4-4i} - 3a^{-b+12} + \sum_{i=1}^{2}(-1)^{5-i}(4-i)a^{-b+12-4i} + 2a^{-b} + a^{-b-8}.$$

Now with the assumption that the result is true for an arbitrary m, we have

$$\begin{aligned}
&\langle\widehat{x_1^b x_2^{m+1}}\rangle \\
= \;& -a^3\langle\widehat{x_1^b x_2^m}\rangle + (-1)^b a^{3b-(m+1)+4} + \sum_{i=1}^{b-3}(-1)^{b+1-i}(i)a^{3b-(m+1)-4i} \\
& +2a^{-b-(m+1)+4} + a^{-b-(m+1)} + a^{-b-(m+1)-4} \\
= \;& \sum_{i=1}^{m-1}(-1)^{b+m+2-i}(i)a^{3(b+m)+3-4i} + (-1)^{b+2}(m-1)a^{3b-m+3} \\
& +m\sum_{i=1}^{b-m-1}(-1)^{b+2-i}a^{3b-m+3-4i} + (-1)^{m+2}(m-1)a^{-b+3m+3} \\
& +\sum_{i=1}^{m-2}(-1)^{m+2-i}(m-i)a^{-b+3m+3-4i} - 2a^{-b-m+7} - a^{-b-m-1} \\
& \quad +(-1)^b a^{3b-(m+1)+4} + \sum_{i=1}^{b-3}(-1)^{b+1-i}(i)a^{3b-(m+1)-4i} \\
& \quad +2a^{-b-(m+1)+4} + a^{-b-(m+1)} + a^{-b-(m+1)-4} \\
= \;& \sum_{i=1}^{(m+1)-1}(-1)^{b+(m+1)+1-i}(i)a^{3(b+(m+1))-4i}
\end{aligned}$$

$$+m\sum_{i=1}^{b-m-1}(-1)^{b+2-i}a^{3b-m+3-4i}+(-1)^{m+2}(m-1)a^{-b+3m+3}$$
$$+\sum_{i=1}^{m-2}(-1)^{m+2-i}(m-i)a^{-b+3m+3-4i}-2a^{-b-m+7}$$

$$+\sum_{i=1}^{b-3}(-1)^{b+1-i}(i)a^{3b-(m+1)-4i}+2a^{-b-(m+1)+4}+a^{-b-(m+1)-4}$$

$$\begin{aligned}= \quad & \sum_{i=1}^{(m+1)-1}(-1)^{b+(m+1)+1-i}(i)a^{3(b+(m+1))-4i}\\ &+\Big[(-1)^{b+1}ma^{3b-m-1}+(-1)^{b}ma^{3b-m-5}+(-1)^{b-1}ma^{3b-m-9}\\ &+\cdots+(-1)^{m+3}ma^{-b+3m+7}+\Big]+(-1)^{m+2}(m-1)a^{-b+3m+3}\\ &+\Big[(-1)^{m+1}(m-1)a^{-b+3m-1}+(-1)^{m}(m-1)a^{-b+3m-5}\\ &+(-1)^{m-1}(m-3)a^{-b+3m-9}+\cdots+(-1)^{4}2a^{-b-m+11}+\Big]\\ &-2a^{-b-m+7}\end{aligned}$$

$$\begin{aligned}&+\Big[\Big((-1)^{b}a^{3b-m-5}+(-1)^{b-1}a^{3b-m-9}+\cdots+(-1)^{m+3}a^{-b+3m+7}\Big)\\ &+\Big((-1)^{m+2}a^{-b+3m+3}+(-1)^{m+1}a^{-b+3m-1}+(-1)^{m}a^{-b+3m-5}\\ &+\cdots+(-1)^{4}a^{-b-m+11}\Big)\Big]+2a^{-b-(m+1)+4}+a^{-b-(m+1)-4}\end{aligned}$$

Now collecting terms of same exponents, we get

$$\begin{aligned}= \quad & \sum_{i=1}^{(m+1)-1}(-1)^{b+(m+1)+1-i}(i)a^{3(b+(m+1))-4i}+(-1)^{b+1}ma^{3b-m-1}\\ &+\Big[+(-1)^{b}(m+1)a^{3b-m-5}+(-1)^{b-1}(m+1)a^{3b-m-9}\\ &+\cdots+(-1)^{m+3}(m+1)a^{-b+3m+7}+\Big]+(-1)^{m+2}(m)a^{-b+3m+3}\\ &+\Big[(-1)^{m+1}(m)a^{-b+3m-1}+(-1)^{m}(m-1)a^{-b+3m-5}\\ &+\cdots+(-1)^{4}3a^{-b-m+11}-2a^{-b-m+7}\Big]\\ &+2a^{-b-(m+1)+4}+a^{-b-(m+1)-4}\end{aligned}$$

which finally, in terms of summation form, is the required result. □

Theorem 3.6. *For any* $b, m \geq 2$,

$$\widehat{\langle x_1^b x_2^m \rangle} = \widehat{\langle x_1^b \rangle}\widehat{\langle x_1^m \rangle}.$$

Proof. Depending on b and m, the proof is divided into three cases: when b, m are even and equal, when b, m are odd and equal, and when b, m are distinct.

Case I. (*When b and m are even and equal.*)
In this case, letting $m = b$, we show that $\widehat{\langle x_1^b x_2^m \rangle} = \widehat{\langle x_1^b \rangle}\widehat{\langle x_1^m \rangle}$. So, we proceed as follows:

$$\widehat{\langle x_1^2 x_2^2 \rangle} = a^8 + 2 + a^{-8} = (-a^4 - a^{-4})(-a^4 - a^{-4}) = \widehat{\langle x_1^2 \rangle}\widehat{\langle x_1^2 \rangle}.$$

Also, we have

$$\begin{aligned} \widehat{\langle x_1^4 x_2^4 \rangle} &= a^{20} - 2a^{16} + 3a^{12} - 2a^8 + 3a^4 - 2 + 2a^{-4} + a^{-12} \\ &= (-a^{10} + a^6 - a^2 - a^{-6})(-a^{10} + a^6 - a^2 - a^{-6}) \\ &= \widehat{\langle x_1^4 \rangle}\widehat{\langle x_1^4 \rangle} \end{aligned}$$

and

$$\begin{aligned} \widehat{\langle x_1^6 x_2^6 \rangle} &= a^{32} - 2a^{28} + 3a^{24} - 4a^{20} + 5a^{16} - 4a^{12} + 5a^8 - 4a^4 \\ &\quad +3 - 2a^{-4} + 2a^{-8} + a^{-16} \\ &= (-a^{16} + a^{12} - a^8 + a^4 - a^0 - a^{-8})(-a^{16} + a^{12} - a^8 \\ &\quad +a^4 - a^0 - a^{-8}) = \widehat{\langle x_1^6 \rangle}\widehat{\langle x_1^6 \rangle}. \end{aligned}$$

Now we assume that the result is true for $b = k$, that is

$$\widehat{\langle x_1^k x_2^k \rangle} = \widehat{\langle x_1^k \rangle}\widehat{\langle x_1^k \rangle}.$$

Since $\widehat{\langle x_1^n \rangle} = -a^{3(n)-2} + a^{3(n)-6} + a^{-2}(\widehat{\langle x_1^{n-2} \rangle})$, we have

$$\begin{aligned} \widehat{\langle x_1^{k+2} \rangle}\widehat{\langle x_1^{k+2} \rangle} &= \big[-a^{3k+4} + a^{3k} + a^{-2}(\widehat{\langle x_1^k \rangle})\big]\big[-a^{3k+4} + a^{3k} \\ &\quad +a^{-2}(\widehat{\langle x_1^k \rangle})\big] \\ &= a^{-4}\big[\widehat{\langle x_1^k \rangle}\big]^2 + a^{6k+8} - 2a^{6k+4} + a^{6k} - 2a^{3k+2}\widehat{\langle x_1^k \rangle} \\ &\quad +2a^{3k-2}\widehat{\langle x_1^k \rangle} \\ &= a^{-4}\big[\widehat{\langle x_1^k \rangle}\big]^2 + a^{6k+8} - 2a^{6k+4} + a^{6k} + 2a^{6k} - 2a^{6k-4} \\ &\quad +2a^{6k-8} - 2a^{6k-12} + \cdots - 2a^{2k+12} + 2a^{2k+8} + 2a^{2k} \\ &\quad -2a^{6k-4} + 2a^{6k-8} - 2a^{6k-12} + 2a^{6k-16} - \cdots + 2a^{2k+8} \\ &\quad -2a^{2k+4} - 2a^{2k-4} \\ &= a^{-4}\big[\widehat{\langle x_1^k \rangle}\big]^2 + a^{6k+8} - 2a^{6k+4} + 3a^{6k} - 4a^{6k-4} + 4a^{6k-8} \\ &\quad -4a^{6k-12} + 4a^{6k-16} - \cdots + 4a^{2k+8} - 2a^{2k+4} + 2a^{2k} \\ &\quad -2a^{2k-4}. \end{aligned} \tag{15}$$

Also

$$\begin{aligned}
\langle \widehat{x_1^{k+2}x_2^{k+2}}\rangle &= a^{-4}\left[\langle \widehat{x_1^k x_2^k}\rangle\right] - \sum_{i=1}^{k-1} i(-1)^{i+1}a^{6k-4i-4} \\
&\quad + \sum_{i=1}^{k-1} i(-1)^{i+1}a^{6k-4i+12} - 2a^{2k-4} \\
&\quad + \sum_{i=k}^{k+1} i(-1)^{i+1}a^{6k-4i+12} - ka^{2k+4} + (k+1)a^{2k} \\
&= a^{-4}\left[\langle \widehat{x_1^k}\rangle\right]^2 - a^{6k-8} + 2a^{6k-12} - 3a^{6k-16} + 4a^{6k-20} \\
&\quad - \cdots - (k-3)(-1)^{k-2}a^{2k+8} - (k-2)(-1)^{k-1}a^{2k+4} \\
&\quad -(k-1)(-1)^k a^{2k} + a^{6k+8} - 2a^{6k+4} + 3a^{6k} - 4a^{6k-4} \\
&\quad +5a^{6k-8} - 6a^{6k-12} + 7a^{6k-16} - 8a^{6k-20} + \cdots \\
&\quad +(k-3)(-1)^{k-2}a^{2k+24} + (k-2)(-1)^{k-1}a^{2k+20} \\
&\quad +(k-1)(-1)^k a^{2k+16} - 2a^{2k-4} + k(-1)^{k+1}a^{2k+12} \\
&\quad +(k+1)(-1)^{k+2}a^{2k+8} - ka^{2k+4} + (k+1)a^{2k} \\
&= a^{-4}\left[\langle \widehat{x_1^k}\rangle\right]^2 + a^{6k+8} - 2a^{6k+4} + 3a^{6k} - 4a^{6k-4} + 4a^{6k-8} \\
&\quad -4a^{6k-12} + 4a^{6k-16} - \cdots + 4a^{2k+8} - 2a^{2k+4} + 2a^{2k} \\
&\quad -2a^{2k-4}. \qquad (16)
\end{aligned}$$

The result now follows from (3.15) and (3.16).

Case II. (*When b and m are odd and equal.*) Similar to Case I.

Case III. (*When b and m are distinct.*)
In order to prove this part let us agree on the terminology:

$$\begin{aligned}
\overline{x}_n &= (-1)^{m+n}a^{3m-(4n-2)}, n = 1,2,\ldots,m-1, \overline{x}_m = -a^{-m-2} \\
\overline{y}_l &= (-1)^{b+l}a^{3b-(4l-2)}, l = 1,2,\ldots,b-1, \overline{y}_b = -a^{-b-2} \\
i &= 1,2,\ldots,m, j = 1,2,\ldots,b; b \geq 2
\end{aligned}$$

$$\begin{aligned}
&\langle \widehat{x_1^b}\rangle\langle \widehat{x_1^m}\rangle \\
= &\sum_{i+j=2}^{m} \overline{x}_i\overline{y}_j + \sum_{i+j=m+1, i\neq m} \overline{x}_i\overline{y}_j + \Bigg[\sum_{i+j=m+2, i\neq m} \overline{x}_i\overline{y}_j + \overline{x}_m\overline{y}_1 \\
&+ \sum_{i+j=m+3, i\neq m} \overline{x}_i\overline{y}_j + \overline{x}_m\overline{y}_2 + \cdots + \sum_{i+j=b, i\neq m} \overline{x}_i\overline{y}_j + \overline{x}_m\overline{y}_{b-m-1}\Bigg]
\end{aligned}$$

$$+\Big[\sum_{i+j=b+1,i\neq 1,m}\overline{x}_i\overline{y}_j+\overline{x}_m\overline{y}_{b-m}\Big]+\Big[\Big(\sum_{i+j=b+2,i\neq 2,m}\overline{x}_i\overline{y}_j+\overline{x}_m\overline{y}_{b-m+1}+\overline{x}_1\overline{y}_b\Big)$$
$$+\Big(\sum_{i+j=b+3,i\neq 3,m}\overline{x}_i\overline{y}_j+\overline{x}_m\overline{y}_{b-m+2}+\overline{x}_2\overline{y}_b\Big)+\cdots$$
$$+\Big(\sum_{i+j=b+m-3,i\neq m-3,m}\overline{x}_i\overline{y}_j+\overline{x}_m\overline{y}_{b-4}+\overline{x}_{m-4}\overline{y}_b\Big)$$
$$+\Big(\overline{x}_{m-1}\overline{y}_{b-1}+\overline{x}_m\overline{y}_{b-3}+\overline{x}_{m-3}\overline{y}_b\Big)+\Big(\overline{x}_m\overline{y}_{b-2}+\overline{x}_{m-2}\overline{y}_b\Big)\Big]$$
$$+\Big(\overline{x}_m\overline{y}_{b-1}+\overline{x}_{m-1}\overline{y}_b\Big)+\overline{x}_m\overline{y}_b$$

Since this agrees with the result of Proposition 3.5, the proof is finished. □

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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