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# Uniformity of dynamic inequalities constituted on time Scales

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**Abstract:** In this article, we present extensions of some well-known inequalities such as Young's inequality and Qi's inequality on fractional calculus of time scales. To find generalizations of such types of dynamic inequalities, we apply the time scale Riemann-Liouville type fractional integrals. We investigate dynamic inequalities on delta calculus and their symmetric nabla results. The theory of time scales is utilized to combine versions in one comprehensive form. The calculus of time scales unifies and extends some continuous forms and their discrete and quantum inequalities. By applying the calculus of time scales, results can be generated in more general form. This hybrid theory is also extensively practiced on dynamic inequalities.

**Keywords:** Fractional calculus, Riemann-Liouville fractional integral, hybrid theory.

## 1. Introduction

The calculus of time scales was initially developed by Stefan Hilger (see [1]). A time scale is an arbitrary nonempty closed subset of the real numbers. The three commonly known examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The time scales calculus is divided into delta calculus, nabla calculus and diamond–alpha calculus. During the last two decades, many researchers have established several dynamic inequalities (see [2–10]). The fundamental work on dynamic inequalities is done by Ravi Agarwal, George Anastassiou, Martin Bohner, Allan Peterson, Donal O'Regan, Samir Saker and many other researchers.

There have been recent developments and refinements of the theory and applications of dynamic inequalities on time scales. From the theoretical perspective, the work provides a coalition and amplification of conventional differential, difference and quantum equations. Moreover, it is a key mechanism in many mathematical, computational, biological, economical and numerical applications.

In this research article, it is accepted that all considerable integrals exist and are finite and $\mathbb{T}$ denotes as usual the time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

## 2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adopted from monographs [6,11]. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that $t$ is right–scattered, while if $\rho(t) < t$, we say that $t$ is left–scattered. Also, if $t < \sup \mathbb{T}$

and $\sigma(t) = t$, then $t$ is called right–dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then $t$ is called left–dense. If $\mathbb{T}$ has a left–scattered maximum $M$, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative $f^{\Delta}$ is defined as follows; Let $t \in \mathbb{T}^k$. If there exists $f^{\Delta}(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there is a neighborhood $U$ of $t$, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then $f$ is said to be delta differentiable at $t$, and $f^{\Delta}(t)$ is called the delta derivative of $f$ at $t$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous), if it is continuous at each right–dense point and there exists a finite left–sided limit at every left–dense point. The set of all rd–continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [6,11].

**Definition 1.** A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of $f$ is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [6,11,12]. If $\mathbb{T}$ has a right–scattered minimum $m$, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called nabla differentiable at $t \in \mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that given any $\epsilon > 0$, there is a neighborhood $V$ of $t$, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be left–dense continuous (ld–continuous), provided it is continuous at all left–dense points in $\mathbb{T}$ and its right–sided limits exist (finite) at all right–dense points in $\mathbb{T}$. The set of all ld–continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$. The next definition is given in [6,11,12].

**Definition 2.** A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of $g$ is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

The following definition is taken from [3,5].

**Definition 3.** For $\alpha \geq 1$, the time scale $\Delta$–Riemann–Liouville type fractional integral for a function $f \in C_{rd}$ is defined by

$$\mathcal{I}_a^{\alpha} f(t) = \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau)\Delta\tau, \tag{1}$$

which is an integral on $[a, t)_{\mathbb{T}}$, see [13] and $h_{\alpha} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise rd–continuous functions, such that $h_0(t, s) = 1$,

$$h_{\alpha+1}(t, s) = \int_s^t h_{\alpha}(\tau, s)\Delta\tau, \ \forall s, t \in \mathbb{T}. \tag{2}$$

Notice that

$$\mathcal{I}_a^1 f(t) = \int_a^t f(\tau)\Delta\tau,$$

which is absolutely continuous in $t \in [a, b]_{\mathbb{T}}$, see [13].

The following definition is taken from [4,5].

**Definition 4.** For $\alpha \geq 1$, the time scale $\nabla$–Riemann–Liouville type fractional integral for a function $f \in C_{ld}$ is defined by

$$\mathcal{J}_a^{\alpha} f(t) = \int_a^t \hat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau) \nabla\tau, \tag{3}$$

which is an integral on $(a, t]_{\mathbb{T}}$, see [13] and $\hat{h}_{\alpha} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise ld–continuous functions, such that $\hat{h}_0(t, s) = 1$,

$$\hat{h}_{\alpha+1}(t, s) = \int_s^t \hat{h}_{\alpha}(\tau, s) \nabla\tau, \ \forall s, t \in \mathbb{T}. \tag{4}$$

Notice that

$$\mathcal{J}_a^1 f(t) = \int_a^t f(\tau) \nabla\tau,$$

which is absolutely continuous in $t \in [a, b]_{\mathbb{T}}$, see [13].

## 3. Dynamic Young's inequality

In order to present our main results, first we give a straightforward proof for an extension of dynamic Young's inequalities by using the time scale $\Delta$–Riemann–Liouville type fractional integral.

**Theorem 5.** *Let $w, f, g \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ be $\Delta$–integrable functions and $h_{\alpha-1}(.,.)$, $h_{\beta-1}(.,.) > 0$. If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequalities hold true for $\alpha, \beta \geq 1$:*

$$\begin{aligned} &\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)||g(x)|^{p-1}\right) \mathcal{I}_a^{\beta}\left(|w(x)||f(x)||g(x)|^{q-1}\right) \leq \frac{1}{p} \mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right) \mathcal{I}_a^{\beta}\left(|w(x)||g(x)|^q\right) \\ &+ \frac{1}{q} \mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right) \mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^q\right), \end{aligned} \tag{5}$$

$$\begin{aligned} &\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right) \mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^{p-1}|g(x)|^{q-1}\right) \leq \frac{1}{p} \mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right) \mathcal{I}_a^{\beta}\left(|w(x)||g(x)|^q\right) \\ &+ \frac{1}{q} \mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^q\right) \mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^p\right) \end{aligned} \tag{6}$$

*and*

$$\begin{aligned} &\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right) \mathcal{I}_a^{\beta}\left(|w(x)||f(x)g(x)|\right) \leq \frac{1}{p} \mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right) \mathcal{I}_a^{\beta}\left(|w(x)||g(x)|^p\right) \\ &+ \frac{1}{q} \mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^q\right) \mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^q\right). \end{aligned} \tag{7}$$

**Proof.** For the proof of inequality (5), we set $\psi = \frac{|f(y)|}{|g(y)|}$ and $\omega = \frac{|f(z)|}{|g(z)|}$, $|g(y)|, |g(z)| \neq 0$, $y, z \in [a, x]_{\mathbb{T}}$, $\forall x \in [a, b]_{\mathbb{T}}$, in the classical Young's inequality $\psi\omega \leq \frac{\psi^p}{p} + \frac{\omega^q}{q}$, $\psi, \omega \geq 0$, we obtain

$$\frac{|f(y)f(z)|}{|g(y)g(z)|} \leq \frac{1}{p} \frac{|f(y)|^p}{|g(y)|^p} + \frac{1}{q} \frac{|f(z)|^q}{|g(z)|^q}. \tag{8}$$

Multiplying inequality (8) by $h_{\alpha-1}(x, \sigma(y)) h_{\beta-1}(x, \sigma(z)) |w(y)w(z)|$, $y, z \in [a, x)_{\mathbb{T}}$, $\forall x \in [a, b]_{\mathbb{T}}$ on both sides and double integrating over $y$ and $z$, respectively, from $a$ to $x$, we get

$$\begin{aligned} &\int_a^x h_{\alpha-1}(x, \sigma(y)) |w(y) f(y)||g(y)|^{p-1} \Delta y \int_a^x h_{\beta-1}(x, \sigma(z)) |w(z) f(z)||g(z)|^{q-1} \Delta z \\ &\leq \frac{1}{p} \left( \int_a^x h_{\alpha-1}(x, \sigma(y)) |w(y)||f(y)|^p \Delta y \right) \left( \int_a^x h_{\beta-1}(x, \sigma(z)) |w(z)||g(z)|^q \Delta z \right) \\ &\quad + \frac{1}{q} \left( \int_a^x h_{\alpha-1}(x, \sigma(y)) |w(y)||g(y)|^p \Delta y \right) \left( \int_a^x h_{\beta-1}(x, \sigma(z)) |w(z)||f(z)|^q \Delta z \right). \end{aligned} \tag{9}$$

Inequality (5) follows from inequality (9).

For the proof of inequality (6), we set $\psi = \frac{|f(y)|}{|f(z)|}$ and $\omega = \frac{|g(y)|}{|g(z)|}$, $|f(z)|, |g(z)| \neq 0, y, z \in [a, x]_{\mathbb{T}}, \forall x \in [a, b]_{\mathbb{T}}$, in the classical Young's inequality $\psi\omega \leq \frac{\psi^p}{p} + \frac{\omega^q}{q}$, $\psi, \omega \geq 0$ and following the same steps used in the proof of inequality (5), we obtain the desired result.

Now, for the proof of inequality (7), we set $\psi = |f(y)g(z)|$ and $\omega = |f(z)g(y)|$, $y, z \in [a, x]_{\mathbb{T}}, \forall x \in [a, b]_{\mathbb{T}}$, in the classical Young's inequality $\psi\omega \leq \frac{\psi^p}{p} + \frac{\omega^q}{q}$, $\psi, \omega \geq 0$ and following the same steps used in the proof of inequality (5), we obtain the desired result. This completes the proof of Theorem 5. □

Next, we give a straightforward proof for an extension of dynamic Young's inequalities by using the time scale $\nabla$–Riemann–Liouville type fractional integral.

**Theorem 6.** *Let $w, f, g \in C_{ld}\left([a, b]_{\mathbb{T}}, \mathbb{R}\right)$ be $\nabla$–integrable functions and $\hat{h}_{\alpha-1}(.,.)$, $\hat{h}_{\beta-1}(.,.) > 0$. If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequalities hold true for $\alpha, \beta \geq 1$:*

$$\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)||g(x)|^{p-1}\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)||g(x)|^{q-1}\right) \leq \frac{1}{p}\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\mathcal{J}_a^{\beta}\left(|w(x)||g(x)|^q\right)$$
$$+\frac{1}{q}\mathcal{J}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^q\right), \tag{10}$$
$$\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^{p-1}|g(x)|^{q-1}\right) \leq \frac{1}{p}\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\mathcal{J}_a^{\beta}\left(|w(x)||g(x)|^q\right)$$
$$+\frac{1}{q}\mathcal{J}_a^{\alpha}\left(|w(x)||g(x)|^q\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^p\right) \tag{11}$$

*and*

$$\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)g(x)|\right) \leq \frac{1}{p}\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\mathcal{J}_a^{\beta}\left(|w(x)||g(x)|^p\right)$$
$$+\frac{1}{q}\mathcal{J}_a^{\alpha}\left(|w(x)||g(x)|^q\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^q\right). \tag{12}$$

**Proof.** Similar to the proof of Theorem 5. □

**Remark 1.** Let $\alpha = \beta = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = n + 1$, $w \equiv 1$, $f(k) = x_k \in [0, +\infty)$ and $g(k) = y_k \in [0, +\infty)$ for $k = 1, 2, \ldots, n$. Then inequalities (5), (6) and (7) become

$$\sum_{k=1}^{n} x_k y_k^{p-1} \sum_{k=1}^{n} x_k y_k^{q-1} \leq \frac{1}{p}\sum_{k=1}^{n} x_k^p \sum_{k=1}^{n} y_k^q + \frac{1}{q}\sum_{k=1}^{n} x_k^q \sum_{k=1}^{n} y_k^p, \tag{13}$$

$$\sum_{k=1}^{n} x_k y_k \sum_{k=1}^{n} x_k^{p-1} y_k^{q-1} \leq \sum_{k=1}^{n} x_k^p \sum_{k=1}^{n} y_k^q \tag{14}$$

and

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \frac{1}{p}\sum_{k=1}^{n} x_k^p \sum_{k=1}^{n} y_k^p + \frac{1}{q}\sum_{k=1}^{n} x_k^q \sum_{k=1}^{n} y_k^q. \tag{15}$$

We give an extension of more dynamic Young's inequalities by using the time scale $\Delta$–Riemann–Liouville type fractional integral.

**Theorem 7.** *Let $w, f, g \in C_{rd}\left([a, b]_{\mathbb{T}}, \mathbb{R}\right)$ be $\Delta$–integrable functions and $h_{\alpha-1}(.,.)$, $h_{\beta-1}(.,.) > 0$. If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequalities hold true for $\alpha, \beta \geq 1$:*

$$\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right)\mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^{\frac{2}{q}}|g(x)|^{\frac{2}{p}}\right) \leq \frac{1}{p}\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\mathcal{I}_a^{\beta}\left(|w(x)||g(x)|^2\right)$$
$$+\frac{1}{q}\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^q\right)\mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^2\right), \tag{16}$$

$$\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^{\frac{2}{p}}|g(x)|^{\frac{2}{q}}\right)\mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^{p-1}|g(x)|^{q-1}\right)\leq\frac{1}{p}\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^{2}\right)\mathcal{I}_a^{\beta}\left(|w(x)||g(x)|^{q}\right)$$
$$+\frac{1}{q}\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^{2}\right)\mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^{p}\right) \tag{17}$$

*and*

$$\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^{\frac{2}{p}}|g(x)|\right)\mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^{\frac{2}{q}}|g(x)|\right)\leq\frac{1}{p}\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^{2}\right)\mathcal{I}_a^{\beta}\left(|w(x)||g(x)|^{p}\right)$$
$$+\frac{1}{q}\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^{q}\right)\mathcal{I}_a^{\beta}\left(|w(x)||f(x)|^{2}\right). \tag{18}$$

**Proof.** For the proof of inequality (16), we set $\psi=|f(y)||g(z)|^{\frac{2}{p}}$ and $\omega=|f(z)|^{\frac{2}{q}}|g(y)|$, $y,z\in[a,x]_{\mathbb{T}}$, $\forall x\in[a,b]_{\mathbb{T}}$, in the classical Young's inequality $\psi\omega\leq\frac{\psi^p}{p}+\frac{\omega^q}{q}$, $\psi,\omega\geq 0$, we obtain

$$|f(y)g(y)||f(z)|^{\frac{2}{q}}|g(z)|^{\frac{2}{p}}\leq\frac{1}{p}|f(y)|^{p}|g(z)|^{2}+\frac{1}{q}|f(z)|^{2}|g(y)|^{q}. \tag{19}$$

Multiplying (19) by $h_{\alpha-1}(x,\sigma(y))h_{\beta-1}(x,\sigma(z))|w(y)w(z)|$, $y,z\in[a,x)_{\mathbb{T}}$, $\forall x\in[a,b]_{\mathbb{T}}$ on both sides and double integrating over $y$ and $z$, respectively, from $a$ to $x$, we get

$$\begin{aligned}&\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)g(y)|\Delta y\int_a^x h_{\beta-1}(x,\sigma(z))|w(z)||f(z)|^{\frac{2}{q}}|g(z)|^{\frac{2}{p}}\Delta z\\ &\leq\frac{1}{p}\left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)|^{p}\Delta y\right)\left(\int_a^x h_{\beta-1}(x,\sigma(z))|w(z)||g(z)|^{2}\Delta z\right)\\ &+\frac{1}{q}\left(\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||g(y)|^{q}\Delta y\right)\left(\int_a^x h_{\beta-1}(x,\sigma(z))|w(z)||f(z)|^{2}\Delta z\right).\end{aligned} \tag{20}$$

Inequality (16) follows from inequality (20).

For the proof of inequality (17), we set $\psi=\frac{|f(y)|^{\frac{2}{p}}}{|f(z)|}$ and $\omega=\frac{|g(y)|^{\frac{2}{q}}}{|g(z)|}$, $|f(z)|,|g(z)|\neq 0$, $y,z\in[a,x]_{\mathbb{T}}$, $\forall x\in[a,b]_{\mathbb{T}}$, in the classical Young's inequality $\psi\omega\leq\frac{\psi^p}{p}+\frac{\omega^q}{q}$, $\psi,\omega\geq 0$ and following the same steps used in the proof of inequality (16), we obtain the desired result.

Now, for the proof of inequality (18), we set $\psi=|f(y)|^{\frac{2}{p}}|g(z)|$ and $\omega=|f(z)|^{\frac{2}{q}}|g(y)|$, $y,z\in[a,x]_{\mathbb{T}}$, $\forall x\in[a,b]_{\mathbb{T}}$, in the classical Young's inequality $\psi\omega\leq\frac{\psi^p}{p}+\frac{\omega^q}{q}$, $\psi,\omega\geq 0$ and following the same steps used in the proof of inequality (16), we obtain the desired result. This completes the proof of Theorem 7. □

Next, we give an extension of more dynamic Young's inequalities by using the time scale $\nabla$–Riemann–Liouville type fractional integral.

**Theorem 8.** *Let $w,f,g\in C_{ld}\left([a,b]_{\mathbb{T}},\mathbb{R}\right)$ be $\nabla$–integrable functions and $\hat{h}_{\alpha-1}(.,.)$, $\hat{h}_{\beta-1}(.,.)>0$. If $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequalities hold true for $\alpha,\beta\geq 1$:*

$$\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^{\frac{2}{q}}|g(x)|^{\frac{2}{p}}\right)\leq\frac{1}{p}\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^{p}\right)\mathcal{J}_a^{\beta}\left(|w(x)||g(x)|^{2}\right)$$
$$+\frac{1}{q}\mathcal{J}_a^{\alpha}\left(|w(x)||g(x)|^{q}\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^{2}\right), \tag{21}$$

$$\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^{\frac{2}{p}}|g(x)|^{\frac{2}{q}}\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^{p-1}|g(x)|^{q-1}\right)\leq\frac{1}{p}\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^{2}\right)\mathcal{J}_a^{\beta}\left(|w(x)||g(x)|^{q}\right)$$
$$+\frac{1}{q}\mathcal{J}_a^{\alpha}\left(|w(x)||g(x)|^{2}\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^{p}\right) \tag{22}$$

*and*

$$\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^{\frac{2}{p}}|g(x)|\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^{\frac{2}{q}}|g(x)|\right) \le \frac{1}{p}\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^2\right)\mathcal{J}_a^{\beta}\left(|w(x)||g(x)|^p\right)$$
$$+\frac{1}{q}\mathcal{J}_a^{\alpha}\left(|w(x)||g(x)|^q\right)\mathcal{J}_a^{\beta}\left(|w(x)||f(x)|^2\right). \tag{23}$$

**Proof.** Similar to the proof of Theorem 7. □

**Remark 2.** Let $\alpha = \beta = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = n+1$, $w \equiv 1$, $f(k) = x_k \in [0, +\infty)$ and $g(k) = y_k \in [0, +\infty)$ for $k = 1, 2, \dots, n$. Then inequalities (16), (17) and (18) become

$$\sum_{k=1}^{n} x_k y_k \sum_{k=1}^{n} x_k^{\frac{2}{q}} y_k^{\frac{2}{p}} \le \frac{1}{p}\sum_{k=1}^{n} x_k^p \sum_{k=1}^{n} y_k^2 + \frac{1}{q}\sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^q, \tag{24}$$

$$\sum_{k=1}^{n} x_k^{\frac{2}{p}} y_k^{\frac{2}{q}} \sum_{k=1}^{n} x_k^{p-1} y_k^{q-1} \le \frac{1}{p}\sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^q + \frac{1}{q}\sum_{k=1}^{n} x_k^p \sum_{k=1}^{n} y_k^2 \tag{25}$$

and

$$\sum_{k=1}^{n} x_k^{\frac{2}{p}} y_k \sum_{k=1}^{n} x_k^{\frac{2}{q}} y_k \le \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n}\left(\frac{y_k^p}{p} + \frac{y_k^q}{q}\right). \tag{26}$$

## 4. Dynamic Qi's inequality

In this section, we give an extension of dynamic Qi's inequalities by using the time scale $\Delta$–Riemann–Liouville type fractional integral.

**Theorem 9.** *Let $w, f, g, h \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\Delta$–integrable functions with $0 < m \le \frac{|f(y)|}{|g(y)|} \le M < \infty$ on $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$ satisfying $|f(y)|^{\frac{1}{p}}|g(y)|^{\frac{1}{q}}|h(y)|^{\frac{1}{r}} = c$, where $c$ is a positive real number. Assume further that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$, $h_{\alpha-1}(.,.) > 0$ and $\alpha \ge 1$.*

*(i) If $p > 0$, $q > 0$, $r < 0$, then*

$$\left\{\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|\right)\right\}^{\frac{1}{q}}\left\{\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|\right)\right\}^{\frac{1}{p}}\left\{\mathcal{I}_a^{\alpha}\left(|w(x)||h(x)|\right)\right\}^{\frac{1}{r}} \ge c\left(\frac{m^{\frac{1}{q^2}}}{M^{\frac{1}{p^2}}}\right)^{-r}. \tag{27}$$

*(ii) If $p < 0$, $q < 0$, $r > 0$, then*

$$\left\{\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|\right)\right\}^{\frac{1}{q}}\left\{\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|\right)\right\}^{\frac{1}{p}}\left\{\mathcal{I}_a^{\alpha}\left(|w(x)||h(x)|\right)\right\}^{\frac{1}{r}} \le c\left(\frac{m^{\frac{1}{q^2}}}{M^{\frac{1}{p^2}}}\right)^{-r}. \tag{28}$$

**Proof.** Case (i). The given condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$ can be rearranged to yield $\frac{1}{\left(-\frac{p}{r}\right)} + \frac{1}{\left(-\frac{q}{r}\right)} = 1$. Applying dynamic Rogers–Hölder's inequality [2] for $-\frac{p}{r} > 1$ and $-\frac{q}{r} > 1$, we get

$$\int_a^x |w(y)||f(y)|^{-\frac{r}{p}}|g(y)|^{-\frac{r}{q}}\Delta y \le \left(\int_a^x |w(y)||f(y)|\Delta y\right)^{-\frac{r}{p}}\left(\int_a^x |w(y)||g(y)|\Delta y\right)^{-\frac{r}{q}}. \tag{29}$$

From (29), we have that

$$\int_a^x |w(y)||f(y)|^{-\frac{r}{p}}|g(y)|^{-\frac{r}{q}}\Delta y \le \left(\int_a^x |w(y)||f(y)|^{-\frac{r}{p}}|f(y)|^{-\frac{r}{q}}\Delta y\right)^{-\frac{r}{p}} \times \left(\int_a^x |w(y)||g(y)|^{-\frac{r}{p}}|g(y)|^{-\frac{r}{q}}\Delta y\right)^{-\frac{r}{q}}. \tag{30}$$

From the given condition, we obtain

$$|f(y)|^{-\frac{r}{p}} \leq (M|g(y)|)^{-\frac{r}{p}}, \quad |g(y)|^{-\frac{r}{q}} \leq m^{\frac{r}{q}}|f(y)|^{-\frac{r}{q}},$$

on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. From the above inequality, it follows that

$$\int_a^x |w(y)||f(y)|^{-\frac{r}{p}}|g(y)|^{-\frac{r}{q}}\Delta y \leq M^{\frac{r^2}{p^2}} m^{-\frac{r^2}{q^2}} \left(\int_a^x |w(y)||g(y)|^{-\frac{r}{p}}|f(y)|^{-\frac{r}{q}}\Delta y\right)^{-\frac{r}{p}}$$
$$\times \left(\int_a^x |w(y)||g(y)|^{-\frac{r}{p}}|f(y)|^{-\frac{r}{q}}\Delta y\right)^{-\frac{r}{q}}. \tag{31}$$

Therefore

$$\int_a^x |w(y)||f(y)|^{-\frac{r}{p}}|g(y)|^{-\frac{r}{q}}\Delta y \leq M^{\frac{r^2}{p^2}} m^{-\frac{r^2}{q^2}} \int_a^x |w(y)||g(y)|^{-\frac{r}{p}}|f(y)|^{-\frac{r}{q}}\Delta y. \tag{32}$$

Again, applying dynamic Rogers–Hölder's inequality on the right–hand side of inequality (32), we obtain

$$\int_a^x |w(y)||f(y)|^{-\frac{r}{p}}|g(y)|^{-\frac{r}{q}}\Delta y \leq M^{\frac{r^2}{p^2}} m^{-\frac{r^2}{q^2}} \left(\int_a^x |w(y)||g(y)|\Delta y\right)^{-\frac{r}{p}} \left(\int_a^x |w(y)||f(y)|\Delta y\right)^{-\frac{r}{q}}. \tag{33}$$

Using the condition that $|f(y)|^{\frac{1}{p}}|g(y)|^{\frac{1}{q}}|h(y)|^{\frac{1}{r}} = c$, where $c$ is a positive real number and $y \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, the inequality (33) becomes

$$\int_a^x c^{-r}|w(y)||h(y)|\Delta y \leq \frac{M^{\frac{r^2}{p^2}}}{m^{\frac{r^2}{q^2}}} \left(\int_a^x |w(y)||g(y)|\Delta y\right)^{-\frac{r}{p}} \left(\int_a^x |w(y)||f(y)|\Delta y\right)^{-\frac{r}{q}}. \tag{34}$$

Taking power $-\frac{1}{r} > 0$ on both sides of inequality (34) and replacing $|w(y)|$ by $h_{\alpha-1}(x,\sigma(y))|w(y)|$, $y \in [a,x)_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, we obtain the desired inequality (27).

The proof of Case (ii) is similar to that of Case (i). This completes the proof of Theorem 9. □

Next, we give an extension of dynamic Qi's inequalities by using the time scale $\nabla$–Riemann–Liouville type fractional integral.

**Theorem 10.** *Let $w, f, g, h \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\nabla$–integrable functions with $0 < m \leq \frac{|f(y)|}{|g(y)|} \leq M < \infty$ on $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$ satisfying $|f(y)|^{\frac{1}{p}}|g(y)|^{\frac{1}{q}}|h(y)|^{\frac{1}{r}} = c$, where $c$ is a positive real number. Assume further that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$, $\hat{h}_{\alpha-1}(.,.) > 0$ and $\alpha \geq 1$.*

(i) *If $p > 0$, $q > 0$, $r < 0$, then*

$$\{\mathcal{J}_a^\alpha(|w(x)||f(x)|)\}^{\frac{1}{q}} \{\mathcal{J}_a^\alpha(|w(x)||g(x)|)\}^{\frac{1}{p}} \{\mathcal{J}_a^\alpha(|w(x)||h(x)|)\}^{\frac{1}{r}} \geq c \left(\frac{m^{\frac{1}{q^2}}}{M^{\frac{1}{p^2}}}\right)^{-r}. \tag{35}$$

(ii) *If $p < 0$, $q < 0$, $r > 0$, then*

$$\{\mathcal{J}_a^\alpha(|w(x)||f(x)|)\}^{\frac{1}{q}} \{\mathcal{J}_a^\alpha(|w(x)||g(x)|)\}^{\frac{1}{p}} \{\mathcal{J}_a^\alpha(|w(x)||h(x)|)\}^{\frac{1}{r}} \leq c \left(\frac{m^{\frac{1}{q^2}}}{M^{\frac{1}{p^2}}}\right)^{-r}. \tag{36}$$

**Proof.** Similar to the proof of Theorem 9. □

**Remark 3.** Let $\alpha = 1$, $\mathbb{T} = \mathbb{R}$, $x = b$, $r = -1$, $c = 1$, $w \equiv 1$ and $f(y), g(y) \in (0, +\infty)$, $\forall y \in [a, b]$. Then inequality (27) reduces to

$$\int_a^b [f(y)]^{\frac{1}{p}} [g(y)]^{\frac{1}{q}} dy \leq \frac{M^{\frac{1}{p^2}}}{m^{\frac{1}{q^2}}} \left( \int_a^b f(y) dy \right)^{\frac{1}{q}} \left( \int_a^b g(y) dy \right)^{\frac{1}{p}}. \tag{37}$$

The inequality (37) can be found in [14].

**Remark 4.** Let $\alpha = 1$, $x = b$, $r = -1$, $c = 1$, $w \equiv 1$ and $f(y), g(y) \in (0, +\infty)$, $\forall y \in [a, b]_{\mathbb{T}}$. Then inequality (27) reduces to

$$\int_a^b [f(y)]^{\frac{1}{p}} [g(y)]^{\frac{1}{q}} \Delta y \leq \frac{M^{\frac{1}{p^2}}}{m^{\frac{1}{q^2}}} \left( \int_a^b f(y) \Delta y \right)^{\frac{1}{q}} \left( \int_a^b g(y) \Delta y \right)^{\frac{1}{p}}. \tag{38}$$

The inequality (38) may be found in [10].

**Corollary 1.** *Let $x_k, y_k, z_k \in (0, +\infty)$ with $0 < m \leq \frac{x_k}{y_k} \leq M < \infty$ for $k \in \{1, 2, \ldots, n\}$ satisfying $x_k^{\frac{1}{p}} y_k^{\frac{1}{q}} z_k^{\frac{1}{r}} = c$, where $c$ is a positive real number. Assume further that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$, $p, q, r \in \mathbb{R} - \{0\}$.*

(i) *If $p > 0$, $q > 0$, $r < 0$, then*

$$\left( \sum_{k=1}^{n} x_k \right)^{\frac{1}{q}} \left( \sum_{k=1}^{n} y_k \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} z_k \right)^{\frac{1}{r}} \geq c \left( \frac{m^{\frac{1}{q^2}}}{M^{\frac{1}{p^2}}} \right)^{-r}. \tag{39}$$

(ii) *If $p < 0$, $q < 0$, $r > 0$, then*

$$\left( \sum_{k=1}^{n} x_k \right)^{\frac{1}{q}} \left( \sum_{k=1}^{n} y_k \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} z_k \right)^{\frac{1}{r}} \leq c \left( \frac{m^{\frac{1}{q^2}}}{M^{\frac{1}{p^2}}} \right)^{-r}. \tag{40}$$

**Proof.** Putting $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = n + 1$ and $w \equiv 1$ in Theorem 9, we obtain the inequalities (39) and (40). □

## 5. Conclusion

Young's inequalities on fractional calculus by means of generalized fractional integrals can be found in [15]. Such inequalities on fractional calculus by Hadamard fractional integral operator can be found in [16]. Motivated by the work, we have obtained dynamic Young's inequalities on fractional calculus of time scales, which has become a significant way in pure and applied mathematics. We have also developed several versions of dynamic Qi's inequalities on fractional calculus of time scales.

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