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# Coupled coincidence and coupled common fixed point theorem in dislocated quasi metric space

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**Abstract:** The objective of this paper is to establish a theorem involving a pair of weakly compatible mappings fulfilling a contractive condition of rational type in the context of dislocated quasi metric space. Besides we proved the existence and uniqueness of coupled coincidence and coupled common fixed point for such mappings. This work offers extension as well as considerable improvement of some results in the existing literature. Lastly, an illustrative example is given to validate our newly proved results.

**Keywords:** Coupled coincidence point, coupled common fixed point, dislocated quasi metric space, pair of weakly compatible mappings.

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## 1. Introduction and Preliminaries

The concept of dislocated metric space was introduced by Hitzler [1] in an effort to generalize the well known Banach contraction principle. Later his work was generalized by Zeyada [2] and many papers covering fixed point results for a single and a pair of mappings satisfying various types of contraction conditions are also published, see [2–4]. Similarly, Bhaskar and Lakshmikantham [5] introduced the concept of coupled fixed point for non-linear contractions in partially ordered metric spaces. After wards, Lakshmikantham and Ćirić [6] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in a complete partially ordered metric space. This area of research has attracted the interest of many researchers and a number of works has been published in different spaces, see [7–10]. Most recently, Mohammad *et al.*, [11] has obtained coupled fixed point finding in the context of dislocated quasi metric space. In this paper, we have established and proved existence and uniqueness of coupled coincidence and coupled common fixed points for a pair of maps in the context of dislocated quasi metric spaces.

## 2. Preliminaries

Now, we present relevant definitions and results that will be retrieved in the sequel and throughout this paper $\Re^+$ will denote the set of non negative real numbers.

**Definition 1.** [1] Let $X$ be a non-empty set and let $d : X \times X \to \Re^+ \cup \{0\}$ be a function satisfying the conditions

(i) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$.

(ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is known as dislocated quasi-metric on $X$ and the pair $(X, d)$ is called a dislocated quasi-metric space.

**Definition 2.** [2] A sequence $\{x_n\}$ in a dislocated quasi metric space $(X, d)$ is said to converge to a point $x \in X$ if and only if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$.

**Definition 3.** [2] A sequence $\{x_n\}$ in a dislocated quasi metric space $(X, d)$ is called a Cauchy sequence if for every $\epsilon > 0$, there exists a positive integer $n_0$ such that for $m, n > n_0$, we have $d(x_n, x_m) < \epsilon$. That is, $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

**Definition 4.** [2] A dislocated quasi metric space is called complete if every Cauchy sequence converges to an element in the same metric space.

**Definition 5.** [12] Let $(X, d)$ be a metric space and $T : X \to X$ be a self-map, then $T$ is said to be a contraction mapping if there exists a constant $k \in [0, 1)$ called a contraction factor, such that $d(Tx, Ty) \leq kd(x, y)$for all $x, y \in X$.

**Definition 6.** [12] Let X be a nonempty set and $T : X \to X$ a self-map. We say that x is a fixed point of T if Tx = x.

**Theorem 7.** *[12] Suppose $(X, d)$ be a complete metric space and $T : X \to X$ be a contraction, then $T$ has a unique fixed point.*

**Definition 8.** [5] An element $(x, y) \in X \times X$ , where $X$ is any non-empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

**Definition 9.** [6] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if F(x, y) = g(x) and F(y, x) = g(y), and $(gx, gy)$ is called coupled point of coincidence.

**Definition 10.** [6] An element $(x, y) \in X \times X$, where $X$ is any non-empty set, is called a coupled common fixed point of the mappings $F : X \times X \to X$ and and $g : X \to X$ if $F(x, y) = g(x) = x$ and $F(y, x) = g(y) = y$.

**Definition 11.** [6] The mappings $F : X \times X \to X$ and $g : X \to X$ are called commutative if $g(F(x, y)) = F(gx, gy)$ for all $x, y \in X$.

**Definition 12.** [6] The mappings $F : X \times X \to X$ and $g : X \to X$ are called w-Compatible if $g(F(x, y)) = F(gx, gy)$ and $g(F(y, x)) = F(gy, gx)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

**Theorem 13.** *[11] Let $(X, d)$ be a complete dislocated quasi-metric space and $T : X \to X$ be a continuous mapping satisfying the following rational type contractive condition*

$$\begin{aligned}
d[T(x,y),T(u,v)] \leq{}& a_1\left[d(x,u)+d(y,v)\right]+a_2\left[d\left(x,T(x,y)\right)+d(u,T(u,v))\right]+a_3\left[d\left(x,T(u,v)\right)+d\left(u,T(x,y)\right)\right]\\
&+a_4\left[\frac{d\left(x,T(x,y)\right)d\left(u,T(u,v)\right)}{d(x,u)+d(y,v)}\right]+a_5\left[\frac{\left(d(x,u)+d(y,v)\right)\times\left(d\left(x,T(x,y)\right)+d\left(u,T(u,v)\right)\right)}{1+d(x,u)+d(y,v)}\right]\\
&+a_6\left[\frac{d\left(x,T(x,y)\right)+d\left(x,T(u,v)\right)}{1+d\left(u,T(u,v)\right)d\left(u,T(x,y)\right)}\right]
\end{aligned}$$

*for all $x, y, u, v \in X$ and $a_1, a_2, a_3, a_4, a_5,$ and $a_6$ are non-negative constants with $2(a_1 + a_2 + a_5) + 4(a_3 + a_6) + a_4 < 1$, then $T$ has a unique coupled fixed point in $X \times X$.*

## 3. Main results

At this stage, we state our theorem and come up with the main findings.

**Theorem 14.** *Let $(X, d)$ be a dislocated quasi-metric space and $T : X \times X \to X$ and $g : X \to X$ be a continuous and commutative mappings satisfying the following rational type contractive condition*

$$\begin{aligned}
d\left((x,y),T(u,v)\right) \leq{}& a_1\left[d(gx,gu)+d(gy,gv)\right]+a_2\left[d\left(gx,T(x,y)\right)+d\left(gu,T(u,v)\right)\right]+a_3\left[d(gx,T(u,v))\right.\\
&\left.+d(gu,T(x,y))\right]+a_4\left[\frac{d(gx,T(x,y))d(gu,T(u,v))}{d(gx,gu)+d(gy,gv)}\right]\\
&+a_5\left[\frac{(d(gx,gu)+d(gy,gv))\times(d(gx,T(x,y))+d(gu,T(u,v)))}{1+d(gx,gu)+d(gy,gv)}\right]\\
&+a_6\left[\frac{d(gx,T(x,y))+d(gx,T(u,v))}{1+d(gu,T(u,v))d(gu,T(x,y))}\right]+a_7\left[\frac{d(gx,T(x,y))d(gu,T(u,v))}{1+d(gx,gu)+d(gu,T(u,v))}\right] \qquad (1)
\end{aligned}$$

*where $x, y, u, v \in X$ and $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \Re^+$ with $2(a_1 + a_2 + a_5) + 4(a_3 + a_6) + a_4 + a_7 < 1$, $T(X \times X) \subseteq g(X)$, and $g(X)$ is complete, then $T$ and $g$ have a unique coupled coincidence point. Moreover, if $T$ and $g$ are weakly compatible, then $T$ and $g$ have unique coupled common fixed point of the form $(u, u)$.*

**Proof.** Let $x_0$ and $y_0 \in X$ and set $gx_1 = T(x_0, y_0)$ and $gy_1 = T(y_0, x_0)$. This is possible since $T(X \times X) \subseteq g(X)$. Proceeding this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $gx_{n+1} = T(x_n, y_n)$ and $gy_{n+1} = T(y_n, x_n)$. Consider $d(gx_n, gx_{n+1}) = d\big[T(x_{n-1}, y_{n-1}), T(x_n, y_n)\big]$. This is in order to show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Now applying (1), we get

$$\begin{aligned}
d(gx_n, gx_{n+1}) &\leq a_1 \left[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\right] + a_2 \left[d(gx_{n-1}, T(x_{n-1}, y_{n-1})) + d(gx_n, T(x_n, y_n))\right] \\
&+ a_3 \left[d(gx_{n-1}, T(x_n, y_n)) + d(gx_n, T(x_{n-1}, y_{n-1}))\right] + a_4 \left[\frac{d(gx_{n-1}, T(x_{n-1}, y_{n-1}))d(gx_n, T(x_n, y_n))}{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}\right] \\
&+ a_5 \left[\frac{(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)) \times \left(d(gx_{n-1}, T(x_{n-1}, y_{n-1})) + d\left(gx_n, T(x_n, y_n)\right)\right)}{1 + d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}\right] \\
&+ a_6 \left[\frac{d(gx_{n-1}, T(x_{n-1}, y_{n-1})) + d(gx_{n-1}, T(x_n, y_n))}{1 + d(gx_n, T(x_n, y_n))d(gx_n, T(x_{n-1}, y_{n-1}))}\right] + a_7 \left[\frac{d(gx_{n-1}, T(x_{n-1}, y_{n-1}))d(gx_n, T(x_n, y_n))}{1 + d(gx_{n-1}, gx_n) + d(gx_n, T(x_n, y_n))}\right].
\end{aligned}$$

At this point, we are going to make use of the definitions of the sequences $\{gx_n\}$ and $\{gy_n\}$ to get

$$\begin{aligned}
d(gx_n, gx_{n+1}) &\leq a_1 \left[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\right] + a_2 \left[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})\right] \\
&+ a_3 \left[d(gx_{n-1}, gx_{n+1}) + d(gx_n, gx_n)\right] + a_4 \left[\frac{d(gx_{n-1}, gx_n)d(gx_n, gx_{n+1})}{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}\right] \\
&+ a_5 \left[\frac{(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)) \times (d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}))}{1 + d(gx_{n-1}, gx_n) + (gy_{n-1}, gy_n)}\right] \\
&+ a_6 \left[\frac{d(gx_{n-1}, gx_n) + d(gx_{n-1}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})d(gx_n, gx_n)}\right] + a_7 \left[\frac{d(gx_{n-1}, gx_n)d(gx_n, gx_{n+1})}{1 + d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})}\right].
\end{aligned}$$

Applying the triangle inequality and the fact that $d(x, y) \geq 0$, we obtain

$$\begin{aligned}
d(gx_n, gx_{n+1}) &\leq a_1 \big[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\big] + a_2 \big[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})\big] \\
&+ a_3 \big[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})\big] + a_4 d(gx_n, gx_{n+1}) \\
&+ a_5 \big[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})\big] + a_6 \big[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n-1}, gx_n) \\
&+ d(gx_n, gx_{n+1})\big] + a_7 d(gx_{n-1}, gx_n).
\end{aligned}$$

Simplification yields

$$\alpha d(gx_n, gx_{n+1}) \leq \beta d(gx_{n-1}, gx_n) + a_1 d(gy_{n-1}, gy_n)$$

where $\alpha = 1 - (a_2 + 2a_3 + a_4 + a_5 + 2a_6)$, and $\beta = a_1 + a_2 + 2a_3 + a_5 + 2a_6 + a_7$. It follows that

$$d(gx_n, gx_{n+1}) \quad \leq \quad \eta d(gx_{n-1}, gx_n) + \theta d(gy_{n-1}, gy_n). \tag{2}$$

where $\eta = \frac{\beta}{\alpha}$ and $\theta = \frac{a_1}{\alpha}$.

Similarly, we can show that

$$d(gy_n, gy_{n+1}) \quad \leq \quad \eta d(gy_{n-1}, gy_n) + \theta d(gx_{n-1}, gx_n). \tag{3}$$

Adding (2) and (3), we get

$$\big[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big] \quad \leq \quad \lambda \big[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\big]. \tag{4}$$

where $\lambda = \eta + \theta$. Similarly, we have

$$\big[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\big] \quad \leq \quad \lambda\big[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})\big].$$

Also

$$\big[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\big] \quad \leq \quad \lambda^2\big[d(gx_{n-3}, gx_{n-2}) + d(gy_{n-3}, gy_{n-2})\big].$$

Continuing this procedure, we obtain

$$\big[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big] \quad \leq \quad \lambda^n\big[d(gx_0, gx_1) + d(gy_0, gy_1)\big]. \tag{5}$$

Since $0 < \lambda < 1$, we have $\lambda^n \to 0$ as $n \to \infty$ and $\big[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big] \to 0$. So $d(gx_n, gx_{n+1}) \to 0$ and $d(gy_n, gy_{n+1}) \to 0$. Applying triangle inequality, using (5), and letting $m > n \geq 1$, it follows that

$$\begin{aligned}
&[d(gx_n, gx_m) + d(gy_n, gy_m)] \leq [d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, gx_{n+3}) + ... + d(gx_{m-1}, gx_m)\\
&\quad + [d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2}) + d(gy_{n+2}, gy_{n+3}) + ... + d(gy_{m-1}, gy_m)\\
&\leq \lambda^n\,[d(gx_0, gx_1) + d(gy_0, gy_1)] + \lambda^{n+1}[d(gx_0, gx_1) + d(gy_0, gy_1) + \lambda^{n+2}[d(gx_0, gx_1) + d(gy_0, gy_1) + ...\\
&\quad + \lambda^{m-1}[d(gx_0, gx_1) + d(gy_0, gy_1) \leq \frac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1)].
\end{aligned}$$

It follows that $\big[d(gx_n, gx_m) + d(gy_n, gy_m)\big] \to 0$ as $n, m \to \infty$ Hence $d(gx_n, gx_m) \to 0$ and $d(gy_n, gy_m) \to 0$ as $n, m \to \infty$. Thus, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. By completeness of $g(X)$ $\exists\ x, y \in g(X)$ such that $\{gx_n\}$ and $\{gy_n\}$ converge to $x$ and $y$ respectively.Now, we prove that $T(x, y) = gx$ and $T(y, x) = gy$. Since $T$ and $g$ are commuting, it follows that

$$ggx_{n+1} = g\big(T(x_n, y_n)\big) = T(gx_n, gy_n) \tag{6}$$

Using (6) and continuity of $T$ and $g$, we have $\lim\limits_{n\to\infty} ggx_n = \lim\limits_{n\to\infty} T(gx_n, gy_n)$, implies $g\left(\lim\limits_{n\to\infty} gx_n\right) = T\left(\lim\limits_{n\to\infty} gx_n, \lim\limits_{n\to\infty} gy_n\right)$. Hence $g(x) = T(x, y)$.

Similarly, we can show that $g(y) = T(y, x)$. Hence, $(gx, gy)$ is coupled point of coincidence of $T$ and $g$. Now, we claim that $(gx, gy)$ is the unique coupled point of coincidence of $T$ and $g$. Suppose, we have another coupled point of coincidence say $(gx_1, gy_1)$ where $(x_1, y_1) \in X^2$ with
$gx_1 = T(x_1, y_1)$ and $gy_1 = T(y_1, x_1)$.
Using (1), we have

$$\begin{aligned}
d(gx, gx) &= d\,[T(x, y), T(x, y)] \leq a_1\,[d(gx, gx) + d(gy, gy)] + a_2\,[d(gx, gx) + d(gx, gx)] + a_3\,[d(gx, gx) + d(gx, gx)]\\
&+ a_4\left[\frac{d(gx, gx)d(gx, gx)}{d(gx, gx) + d(gy, gy)}\right] + a_5\left[\frac{[d(gx, gx) + d(gy, gy)][d(gx, gx) + d(gx, gx)]}{1 + d(gx, gx) + d(gy, gy)}\right] + a_6\left[\frac{d(gx, gx) + d(gx, gx)}{1 + d(gx, gx)d(gx, gx)}\right]\\
&+ a_7\left[\frac{d(gx, gx)d(gx, gx)}{1 + d(gx, gx) + d(gx, gx)}\right].
\end{aligned}$$

Now, we have

$$\begin{aligned}
d(gx, gx) &\leq a_1\big[d(gx, gx) + d(gy, gy)\big] + a_2\big[d(gx, gx) + d(gx, gx)\big] + a_3\big[d(gx, gx) + d(gx, gx)\big] + a_4 d(gx, gx)\\
&+ a_5\big[d(gx, gx) + d(gx, gx)\big] + a_6\big[d(gx, gx) + d(gx, gx)\big] + a_7 d(gx, gx).
\end{aligned}$$

It follows that

$$d(gx, gx) \leq \phi d(gx, gx) + a_1 d(gy, gy) \tag{7}$$

where $\phi = a_1 + 2a_2 + 2a_3 + a_4 + 2a_5 + 2a_6 + a_7$. Similarly

$$d(gy, gy) \leq \phi d(gy, gy) + a_1 d(gx, gx). \tag{8}$$

Adding (7) and (8), we get

$$\left[d(gx,gx)+d(gy,gy)\right] \leq \psi\left[d(gx,gx)+d(gy,gy)\right].$$

where $\psi = \phi + a_1$. This is possible only when $d(gx,gx)+d(gy,gy) = 0$ since $\psi < 1$ which implies that $d(gx,gx) = 0$ and $d(gy,gy) = 0$. Similarly $d(gx_1,gx_1) = 0$ and $d(gy_1,gy_1) = 0$. Now, we shall show the uniqueness of the coupled point of coincidence of $T$ and $g$. For this task, we consider $d(gx,gx_1)$. Using (1), we have

$$\begin{aligned}
d(gx,gx_1) &= d\left[T(x,y),T(x_1,y_1)\right] \leq a_1\left[d(gx,gx_1)+d(gy,gy_1)\right] + a_2\left[d(gx,T(x,y))+d(gx_1,T(x_1,y_1))\right] \\
&+ a_3\left[d(gx,T(x_1,y_1))+d(gx_1,T(x,y))\right] + a_4\left[\frac{d(gx,T(x,y))d(gx_1,T(x_1,y_1))}{d(gx,gx_1)+d(gy,gy_1)}\right] \\
&+ a_5\left[\frac{[d(gx,gx_1)+d(gy,gy_1)]\times[d(gx,T(x,y))+d(gx_1,T(x_1,y_1))]}{1+d(gx,gx_1)+d(gy,gy_1)}\right] \\
&+ a_6\left[\frac{d(gx,T(x,y))+d(gx,T(x_1,y_1))}{1+d(gx_1,T(x_1,y_1))d(gx_1,T(x_1,y_1))}\right] + a_7\left[\frac{d(gx,T(x,y))d(gx_1,T(x_1,y_1))}{1+d(gx,gx_1)+d(gx_1,T(x_1,y_1))}\right].
\end{aligned}$$

Using the fact that $gx = T(x,y)$ and $gx_1 = T(x_1,y_1)$, we have

$$\begin{aligned}
d(gx,gx_1) &\leq a_1\left[d(gx,gx_1)+d(gy,gy_1)\right] + a_2\left[d(gx,gx)+d(gx_1,gx_1)\right] + a_3\left[d(gx,gx_1)+d(gx_1,gx)\right] \\
&+ a_4\left[\frac{d(gx,gx)d(gx_1,gx_1)}{d(gx,gx_1)+d(gy,gy_1)}\right] + a_5\left[\frac{[d(gx,gx_1)+d(gy,gy_1)][d(gx,gx)+d(gx_1,gx_1)]}{1+d(gx,gx_1)+d(gy,gy_1)}\right] \\
&+ a_6\left[\frac{d(gx,gx)+d(gx,gx_1)}{1+d(gx_1,gx_1)d(gx_1,gx_1)}\right] + a_7\left[\frac{d(gx,gx)d(gx_1,gx_1)}{1+d(gx,gx_1)+d(gx_1,gx_1)}\right].
\end{aligned}$$

Thus, we have

$$d(gx,gx_1) \leq (a_1+a_3+a_6)d(gx,gx_1) + a_1 d(gy,gy_1) + (a_3+a_6)d(gx_1,gx),$$

implies

$$(1-(a_1+a_3+a_6))d(gx,gx_1) \quad \leq \quad a_1 d(gy,gy_1) + (a_3+a_6)d(gx_1,gx). \tag{9}$$

Similarly

$$(1-(a_1+a_3+a_6))d(gy,gy_1) \leq a_1 d(gx,gx_1) + (a_3+a_6)d(gy_1,gy). \tag{10}$$

Adding (9) and (10) and then simplifying, we get

$$\left[d(gx,gx_1)+d(gy,gy_1)\right] \leq \omega\left[d(gx_1,gx)+d(gy_1,gy)\right] \tag{11}$$

where $\omega = \left[\frac{a_3+a_6}{1-(2a_1+a_3+a_6)}\right]$. Similarly

$$\left[d(gx_1,gx)+d(gy_1,gy)\right] \leq \omega\left[d(gx,gx_1)+d(gy,gy_1)\right]. \tag{12}$$

Adding (11) and (12), we get

$$\left[d(gx_1,gx)+d(gy_1,gy)+d(gx,gx_1)+d(gy,gy_1)\right] \leq \omega\left[d(gx_1,gx)+d(gy_1,gy)+d(gx,gx_1)+d(gy,gy_1)\right]. \tag{13}$$

So, $\left[d(gx_1,gx)+d(gy_1,gy)+d(gx,gx_1)+d(gy,gy_1)\right] = 0$, since $\omega < 1$. It follows that $d(gx_1,gx) = d(gy_1,gy) = d(gx,gx_1) = d(gy,gy_1) = 0$. Now, applying Definition (1), we get $gx_1 = gx$ and $gy_1 = gy$. Thus, $(gx,gy)$ is the unique coupled point of coincidence of $T$ and $g$.

Next, we show that $gx = gy$.

$$\begin{aligned} d(gx,gy) &= d\big[T(x,y),T(y,x)\big] \leq a_1\big[d(gx,gy)+d(gy,gx)\big] + a_2\Big[d\big(gx,T(x,y)\big)+d\big(gy,T(y,x)\big)\Big] \\ &+ a_3\Big[d\big(gx,T(y,x)\big)+d\big(gy,T(x,y)\big)\Big] + a_4\left[\frac{d\big(gx,T(x,y)\big)d\big(gy,T(y,x)\big)}{d(gx,gy)+d(gy,gx)}\right] \\ &+ a_5\left[\frac{\Big[d(gx,gy)+d(gy,gx)\Big]\Big[d\big(gx,T(x,y)\big)+d\big(gy,T(y,x)\big)\Big]}{1+d(gx,gy)+d(gy,gx)}\right] \\ &+ a_6\left[\frac{d\big(gx,T(x,y)\big)+d\big(gx,T(y,x)\big)}{1+d\big(gy,T(y,x)\big)d\big(gy,T(y,x)\big)}\right] + a_7\left[\frac{d\big(gx,T(x,y)\big)d\big(gy,T(y,x)\big)}{1+d(gx,gy)+d\big(gy,T(y,x)\big)}\right]. \end{aligned}$$

Using (1) and the fact that $gx = T(x,y)$ and $gy = T(y,x)$, we have

$$\begin{aligned} d(gx,gy) &\leq a_1\big[d(gx,gy)+d(gy,gx)\big] + a_2\big[d(gx,gx)+d(gy,gy)\big] + a_3\big[d(gx,gy)+d(gy,gx)\big] \\ &+ a_4\left[\frac{d(gx,gx)d(gy,gy)}{d(gx,gy)+d(gy,gx)}\right] + a_5\left[\frac{\Big[d(gx,gy)+d(gy,gx)\Big]\Big[d(gx,gx)+d(gy,gy)\Big]}{1+d(gx,gy)+d(gy,gx)}\right] \\ &+ a_6\left[\frac{d(gx,gx)+d(gx,gy)}{1+d(gy,gy)d(gy,gy)}\right] + a_7\left[\frac{d(gx,gx)d(gy,gy)}{1+d(gx,gy)+d(gy,gy)}\right]. \end{aligned}$$

Thus, we have

$$d(gx,gy) \quad\leq\quad \sigma d(gy,gx), \tag{14}$$

where $\sigma = \left[\frac{a_1+a_3}{1-(a_1+a_3+a_6)}\right]$. Similarly, we can show that

$$d(gy,gx) \quad\leq\quad \sigma d(gx,gy). \tag{15}$$

Adding (14) and (15), we have $\big[d(gx,gy)+d(gy,gx)\big] \leq \sigma\big[d(gx,gy)+d(gy,gx)\big]$. Since $\sigma < 1$, the above inequality is only possible if $d(gx,gy) = d(gy,gx) = 0$. That is, $gx = gy$. Now, we show that $T$ and $g$ have coupled common fixed point. To do so, first let $u = gx = T(x,y)$. Due to the fact that $T$ and $g$ are weakly compatible, we have $gu = g(gx) = gT(x,y) = T(gx,gy) = T(u,u)$. Hence $(gu,gu)$ is a coupled point of coincidence and $(u,u)$ is a coupled coincidence point of $T$ and $g$. Applying the uniqueness property of coupled point of coincidence of $T$ and $g$, we get $gu = u = gx = gy$. Therefore $T(u,u) = gu = u$. That is $(u,u)$ is a coupled common fixed point of $T$ and $g$. Now it remains to show the uniqueness of a coupled common fixed point of $T$ and $g$. Assume, we have another coupled common fixed point of $T$ and $g$ say $(u_1,u_1) \in X^2$. It follows that $u_1 = gu_1 = T(u_1,u_1)$. Hence $(gu,gu)$ and $(gu_1,gu_1)$ are two coupled points of coincidence of $T$ and $g$. But due to the uniqueness of coupled point of coincidence, we get $gu = gu_1$ and so $u_1 = T(u_1,u_1) = T(u,u) = u$. Therefore $(u,u)$ is the unique coupled common fixed point of $T$ and $g$. $\square$

**Remark 1.** If we take $g = I$ (the identity map) and $a_7 = 0$ in Theorem 14, we get Theorem 13 of [11].

The following example supports our main theorem.

**Example 1.** Let $X = [0,1)$ and $d : X \times X \to \Re^+$ be defined by $d(x,y) = |x-y| + |y|$ for all $x,y \in X$. Then $(X,d)$ is $dq$-metric space. We define the functions $T : X \times X \to X$ and $g : X \to X$ by

$$gx = \begin{cases} \frac{1}{3}x & \text{if } \ 0 \leq x < \frac{9}{10}, \\ \frac{3}{10} & \text{if } \ \frac{9}{10} \leq x < 1, \end{cases}$$

and

$$T(x,y)=\begin{cases}\frac{x+y}{27} & \text{if } \ 0\le x,y<\frac{9}{10},\\ \frac{1}{30}y & \text{if } \ \frac{9}{10}\le x<1 \text{ and } 0\le y<\frac{9}{10},\\ \frac{1}{30}x & \text{if } \ \frac{9}{10}<y<1 \text{ and } 0\le x<\frac{9}{10},\\ \frac{1}{15} & \text{if } \ \frac{9}{10}\le x<1 \text{ and } \frac{9}{10}\le y<1.\end{cases}$$

Clearly $T$ and $g$ are continuous, $T(X\times X)\subseteq g(X)$, and $g(X)$ is complete. Following four cases will arise for $x$, $u$, $v$, and $y$;

Case (1): $0\le x,u,y,v<\frac{9}{10}$.
Case (2): $\frac{9}{10}\le x,u<1$ and $0\le y,v<\frac{9}{10}$.
Case (3): $\frac{9}{10}<y,v<1$ and $0\le x,u<\frac{9}{10}$.
Case (4): $\frac{9}{10}\le x,u<1$ and $\frac{9}{10}\le y,v<1$.

**Case 1**: For $0\le x,u,y,v<\frac{9}{10}$, we have

$$\begin{aligned} d[T(x,y),T(u,v)] &= d\left(\frac{x+y}{27},\frac{u+v}{27}\right)\\ &= \left|\frac{x+y}{27}-\frac{u+v}{27}\right|+\left|\frac{u+v}{27}\right|\\ &= \left|\frac{x}{27}+\frac{y}{27}-\frac{u}{27}-\frac{v}{27}\right|+\left|\frac{u}{27}+\frac{v}{27}\right|\\ &\le \left|\frac{x}{27}-\frac{u}{27}\right|+\left|\frac{y}{27}-\frac{v}{27}\right|+\left|\frac{u}{27}\right|+\left|\frac{v}{27}\right|\\ &= \frac{1}{9}\left[\left(\left|\frac{x}{3}-\frac{u}{3}\right|+\left|\frac{u}{3}\right|\right)+\left(\left|\frac{y}{3}-\frac{v}{3}\right|+\left|\frac{v}{3}\right|\right)\right]\\ &\le \frac{1}{9}\left[d(gx,gu)+d(gy,gv)\right]\\ &\le \frac{2}{9}\left[d(gx,gu)+d(gy,gv)\right]. \end{aligned}$$

Similarly, for Cases (2) to (4), we obtain

$$d\big[T(x,y),T(u,v)\big]\le\frac{2}{9}\big[d(gx,gu)+d(gy,gv)\big].$$

Hence all the conditions of Theorem 13 are satisfied with $a_1=\frac{2}{9}, a_2=\frac{1}{120}, a_3=\frac{1}{64}, a_4=\frac{1}{80}, a_5=\frac{1}{100}, a_6=\frac{1}{128}$, and $a_7=\frac{1}{32}$. Therefore, $T$ and $g$ have unique coupled point of coincidence and unique coupled common fixed point which are $(g0,g0)$ and $(0,0)$ respectively. This is due to the fact that $gT(0,0)=T(g0,g0)=T(0,0)=0$.

## 4. Conclusion

In 2018, Mohammed established the existence of coupled fixed point for mapping satisfying certain rational type contraction condition in a complete dislocated quasi metric space. In this paper, we explored the properties of dislocated quasi-metric spaces and also discuss the difference between metric space and dislocated metric space. We established and proved existence of coupled coincidence point and existence and uniqueness of coupled common fixed point theorem for a pair of maps $T$ and $g$ in the setting of dislocated quasi metric spaces. Also, we provided an example in support of our main result. Our work extended coupled fixed point result to common coupled fixed point result. The presented theorem extends and generalizes several well-known comparable results in literature.

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