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A Tutorial Exposition of Various Methods for Analyzing Capacitated Networks

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Authors' contributions

This work was carried out in collaboration between both authors. Author AMAR envisioned and designed the study, performed the symbolic and numerical analysis, managed literature search and substantially edited the entire manuscript. Author OMA contributed to the symbolic and numerical analysis, prepared the maps, drew the various figures, wrote the first draft of the manuscript and contributed to literature search. Both authors read and approved the final manuscript.

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Abstract

In order to assess the performance indexes of some practical systems having fixed channel capacities, such as telecommunication networks, power transmission systems or commodity pipeline systems, we propose various types of techniques for analyzing a capacitated network. These include Karnaugh maps, capacity-preserving network reduction rules associated with delta-star transformations, and a generalization of the max-flow min-cut theorem. All methods rely on recognizing the network capacity function as a random pseudo-Boolean function of link successes; a fact that allows the expected value of this function to be easily obtainable from its sum-of-products expression. This network capacity has certain advantages for representation of nonbinary discrete random functions, mostly employed in the analysis of flow networks. Five tutorial examples demonstrate the afore-mentioned methods and illustrate their computational advantages over the exhaustive state enumeration method.

Keywords: Capacitated networks; map method; reduction rule; max-flow min-cut theorem; star-delta transformation; pseudo-switching function.

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1 Introduction

In classical source-to-terminal reliability analysis, a frequently used assumption is that the network under study can be modeled by a probabilistic two-state graph, with the network being successful when there exists at least one path originating from the source node and ending at the terminal node. According to this point of view, reliability is thus taken as being solely a matter of connectivity, a simplification that does not seem to reflect, to a reasonable extent, the nature of the problem, or to capture the essence of most real-life networks.

In modern society, we have telecommunication systems, electric power transmission and distribution systems, oil/gas production systems and transportation systems, all of which are physical systems that play vital roles. These systems, in fact, might be appropriately modeled as being capacitated-flow networks having independent edge capacities, that are limited real-valued random variables. Usually, the modelling of a network of this type is attained by the use of a stochastic graph G = (V, E) with E and V being sets of nodes (vertices) and branches (edges) of G, where we can distinguish a particular set $K \subseteq V$[1].

A very important special case arises when the set K is an ordered set of just two nodes: a source node represented by s and a terminal node denoted by t. There are two main parameters that are usually used in quantifying the network performance. These are (a) Network reliability that basically measures connectivity in a probabilistic sense as it equates to the probability of definite connections on directed or undirected general or special graphs, with dependent or independent components (nodes or arcs) existing in G among the nodes in K (and from s and t in the particular st case) [1-9].

Recently, Ching and Hsu [10] proposed a methodology for estimation of the source-to-terminal reliability as connectivity in actual networks; with the critical links being ranked. The two extreme situations are those of (a) the afore-mentioned st case when K contains only two nodes, the source s and the destination t, or (b) when K contains all nodes of the graph for which (K=V), typically depicted as the overall reliability case [11].

Generally, the desired level of attention is not being given to the capacity constraints in the different links and the overall requirements in flow for the network; (b) Network s-t whose capacity is equal to the maximum flow that is traversable to the terminal node from source node with no violation of branch capacity and assuming all branches are functioning well [12-18]. Under such a scheme of deterministic modeling, there is obviously deliberate implicit ignoring of the failure probabilities of both communication links and nodes.

Aggarwal [19], Trstensky and Bowron [20], Ramírez and Gebre [21], Yeh [22], Patra and Misra [23], Fusheng [24], El Khadiri and Yeh [25], Kabadurmus and Smith [26] and Cancela, et al. [27] suggested methods for defining a composite performance index related to any network, that integrates the two afore-mentioned connectivity and capacity aspects. Moreover, Lin et al. [28,29] presented an algorithm for evaluation of capacitated flow network in terms of minimal path sets or minimal cut-sets.

In all methods it is always important to bear in mind that connection between the source and the sink nodes is an important condition for ensuring success in communication network operation. This condition, however, is not to be taken as a sufficient one. It is also crucial that the s-t capacitated connection success should not only establish just a mere s-t connection, but it should also ensure the availability of at least the required s-t capacity.

This paper presents a tutorial exposition of various methods for analyzing capacitated networks. The first method involves a map procedure resulting into having a simple symbolic expressions of performance indexes. However, there can only be a manual application of this technique for cases involving small networks [30-42].

The second method involves application of network transformations or reduction rules, which are being implemented in such a way as to ensure the preservation of the network capacity function. For the application of these rules in series-parallel subnetworks, no appreciable difficulty is encountered. However, whenever a network is sp-complex (i.e., irreducible to a series-parallel graph since bridging branches are involved), there is significant difficulty that is to be addressed through more involved technique such as a function expansion or network decomposition [30,43].

There is also some possibility for easy analysis of a large class of complex networks when these networks are reduced to equivalent series-parallel ones by using the delta-star transformations. Rushdi [44,45] presented a set of star-delta and delta-star transformations that are used in flow networks which are given in terms of the concept of pseudo-Boolean (switching) functions. The concept presents some advantages with regard to the study of discrete random functions that come up when networks are being analyzed. This is in addition to where preservation of s-t capacity function for flow network. In a third method, there is generalization of “Max-Flow Min-Cut Theorem” [12-18] for network states **X** in addition to the ideal state (**X**=1). This is believed to be a very fast technique if we have minimal cut-sets [46], and where possibly minimal paths [46] have been identified.

The remainder of this paper is structured as follows. Section 2 presents the underlying assumptions for our model, the notation used as well as some useful nomenclature.

Section 3 reviews the concept of the algebraic decomposition formula which is a pseudo-switching function that used to obtain the general capacity function for the network.

Section 4 shows how the Karnaugh map is conveniently used to represent a pseudo switching function which is a very powerful manual tool that provides pictorial insight about the various functional properties and procedures. Section 5 explains a reduction rule technique for series-parallel connections and provides examples to illustrate this technique.

In Section 6 and Section 7, clarify the general transformation conditions and demonstrate by an example delta-star transformation which preserve the source-to-terminal (s-t) capacity function in a flow network. Moreover, Section 8 extends this work by presenting one of the crucial and old technique in capacitated network which is “Max-Flow Min-Cut Theorem”. One more example, that is given in this section, shows the equivalence of the results between this technique and afore-mentioned one. Section 9 concludes the paper.

2 Assumptions, Notation, & Nomenclature

2.1 Assumptions

(1) The physical network considered is modeled as a linear graph consisting of (a) transmission links of imperfect reliabilities and limited capacities and (b) nodes which are perfectly reliable and have unconstrained capacities.

(2) Originally, each link in the network has two states, a successful state and an unsuccessful one. Link successes are statistically independent. This assumption does not extend to ‘equivalent' links to be introduced in the network, which can be of multistate natures and statistically dependent.

(3) Certain values are assigned to each link(i,j) for its reliability p_{ij} and capacity c_{ij} , where $0 \leq p_{ij} \leq 1$, $c_{ij} \geq 0$. The link capacity sets an upper bound on link flow in either direction.

(4) Every link in the network is directed. A bidirectional link is replaced by two directed links in antiparallel whose failures are completely dependent. These two links have equivalent reliabilities. However, they perhaps have different capacities.

2.2 Notation

n Number of branches (edges or links) in the logic diagram of the network. The same symbol is used also to depict the common node of a star.

$X_i, \overline{X}_i$ Indicator variables for successful and unsuccessful operation of branch i. These are binary random variables that take only one of the two discrete real values 0 and 1; $X_i = 1$ and $\overline{X}_i = 0$ if i is functioning, and $X_i = 0$ and $\overline{X}_i = 1$ if i is failed. For a bidirectional branch ij, the anti-parallel successes are the same $X_{ij} = X_{ji}$.

$S, \overline{S}$ Indicator variables for successful and unsuccessful operation of the system; called system success and system failure, respectively. Successful operation can be equivalent to connectivity, or to the satisfaction of a certain flow requirement [19,46,47].

p_i, q_i Reliability and unreliability of branch: $p_i \equiv \Pr\{X_i = 1\}, q_i \equiv \Pr\{\overline{X}_i = 1\} = 1 - p_i$. Both $p_i \; and \; q_i$ are real values in the closed real interval [0.0,1.0].

R, U Network reliability and unreliability; $R = \Pr\{\mathrm{S} = 1\} = E\{S\}$, $U = \Pr\{\overline{S} = 1\} = 1.0 - R$, $0.0 \leq R$, $U \leq 1.0$.

c_i Flow capacity of branch; $c_i \geq 0$.

$\boldsymbol{X}, \boldsymbol{p}, \boldsymbol{c}$ n-dimensional vectors of branch successes, reliabilities and capacities:

$\boldsymbol{X} = (X_1 X_2 \dots X_n)^T; \boldsymbol{p} = (p_1 p_2 \dots p_n)^T; \boldsymbol{c} \equiv (c_1 c_2 \dots c_n)^T$.

T A superscript that implies the transpose of a matrix.

$\boldsymbol{X}_k$ State k of the network, denoted by a particular value of the n-dimensional vector **X**, $k = 0,1,2, \dots, 2^n - 1$.

$C_{ij}(\boldsymbol{X})$ Capacity function of (i,j) which is the maximum flow interconnection from i to j in state $\boldsymbol{X}$ that does not violate branch capacities, $C_{ij}(\boldsymbol{X}) \geq 0$. For an original (i,j) : $C_{ij} = c_{ij} X_{ij}$. Since $\boldsymbol{X}$ is a switching random vector, $C_{ij}(\boldsymbol{X})$ is a discrete random variable of a probability mass function (pmf) of no more than 2^n distinct values.

C_{ij}^T Terminal- pair capacity function from node i to node j; $C_{ij}^T \geq 0$.

C_{ijmax} Maximum Capacity function of the (i,j) edge; in the ideal case when all branches are functioning, $C_{ijmax} = C_{ij}(1)$.

D_{ij} Capacity of interconnection from node i to node j in a delta; $D_{ij} \geq 0$. For an original(i,j) : $D_{ij} = d_{ij} X_{ij}$.

S_{ij} Capacity of interconnection from node i to node j in a star; $S_{ij} \geq 0$. For an original (i,j) : $S_{ij} = s_{ij} X_{ij}$.

(i, j) A directed branch or edge from node i to node j. If two or more such branches exist, they are distinguished by superscripts.

s, t Source, terminal node

$C_{ij}(\boldsymbol{X}|1_l), C_{ij}(\boldsymbol{X}|0_l)$ The function $C_{ij}(\boldsymbol{X})$ when X_l is set to 1 or 0. Meanings of $C_{ij}(\boldsymbol{X}|1_l, 1_m), etc.$ follow similarity.

2.3 Nomenclature

A Boolean (Switching) function $S(\boldsymbol{X})$: A mapping $\{0,1\}^n \rightarrow \{0,1\}$, i.e., $S(\boldsymbol{X})$ is any one particular assignment of the two functional values (0 or 1) for all possible 2^n values of $\boldsymbol{X}$ [32-34,48-50].

Pseudo- Boolean (Switching) function $C(\boldsymbol{X})$: A mapping $\{0,1\}^n \rightarrow R$ where R is the field of real numbers, i.e. $C(\boldsymbol{X})$ is an assignment of a real number for each of the possible 2^n values of $\boldsymbol{X}$ [51-53].

Multiaffine function of n variables $\boldsymbol{R(p_1, p_2 \dots, p_n)}$: An algebraic function which is a first-degree polynomial in each of its variables, i.e. if fixed values are given to any $(n-1)$ variables, the function reduces to a first-degree polynomial in the remaining variable. Examples of multi-affine functions involve:

1. Definite algebraic functions such as
   (a) System reliability/unreliability as a function of component reliability/unreliability [54,55].
   (b) System availability/unavailability [55-57].
2. Pseudo-Boolean (switching) functions [51-53] such as source-to-terminal capacity or the squared capacity as a function of link successes.

3 Capacity and Its Mean

The function $C_{ij}(\boldsymbol{X})$ as an expression of the source-to-terminal capacity function of element successes is a real valued function of binary arguments. Therefore, the function $C_{ij}(\boldsymbol{X})$ conforms to the rules of the algebraic decomposition relation of a pseudo-Boolean (switching) function.

$$\begin{aligned} C_{ij}(\boldsymbol{X}) &= \overline{X}_l\, C_{ij}(\boldsymbol{X}|O_l) + X_l C_{ij}(\boldsymbol{X}|1_l) \\ &= (1-X_l) C_{ij}(\boldsymbol{X}|0_l) + X_l\, C_{ij}(\boldsymbol{X}|1_l) \\ &= C_{ij}(\boldsymbol{X}|O_l) + \left[\, C_{ij}(\boldsymbol{X}|1_l) - C_{ij}(\boldsymbol{X}|O_l)\right] X_l\,,\ l = 1,2,\dots,n \end{aligned} \tag{1}$$

Equation (1) can be validated through proof by induction of all cases or values of $\boldsymbol{X}$, viz., $\{\boldsymbol{X}|0_l\}$ and $\{\boldsymbol{X}|1_l\}$. This decomposition relation of $C_{ij}(\boldsymbol{X})$ can be used to deduce many properties of it as a pseudo-switching function, including, in particular, its being a multi-affine function, and the fact that it can be expressed as a sum-of-products form, where the term ‘sum’ here refers to its genuine meaning of real addition. Moreover, $C_{ij}(\boldsymbol{X})$ can be viewed as an assignment of a real number for each of the possible 2^n values of $\boldsymbol{X}$. Subsequently, $C_{ij}(\boldsymbol{X})$ can be accurately represented in the context of a Karnaugh map or a truth table of the usual Boolean combinations of the input domain, but of entries that are real elements rather than binary values. The mean (expected) value of the random function $C_{ij}(\boldsymbol{X})$, when written in sum-of-products form, equates to;

$$E\{C_{ij}(\boldsymbol{X})\} = E\{C_{ij}\}(\boldsymbol{p})\,,$$

and can be directly obtained (on a one-to-one basis) from $C_{ij}(\boldsymbol{X})$(s-o-p) by introducing the component means $p_l = \mathrm{E}\{X_l\}$ and $q_l = E\{\overline{X}_l\}$, in place of the corresponding Boolean arguments X_l, and $\overline{X}_l$, namely,

$$C_{ij}(\boldsymbol{X})(\text{s-o-p}) \xleftrightarrow{\{X_l, \overline{X}_l\} \leftrightarrow \{p_l, q_l\}} E\{C_{ij}\}(\boldsymbol{p})(\text{s-o-p}) \tag{2}$$

Another subtle replacement that is implicit in (2) pertains to substituting arithmetic multiplication in the R.H.S. for the logical multiplication in the L.H.S., a substitution that is not apparent since both operations are represented by juxta-positioning. Equation (2) is an immediate result of the condition that the mean of a sum is the sum of means and that the $X_l's$ are statistically independent. It is important to note that the capacity $C_{ij}(\boldsymbol{X})$ and its square $C_{ij}^2(\boldsymbol{X})$ are both pseudo-switching functions. Thus, to readily convert $C_{ij}^2(\boldsymbol{X})$ into its mean, $C_{ij}^2(\boldsymbol{X})$ can be represented in s-o-p form, namely

$$C_{ij}^2(\boldsymbol{X})(\text{s-o-p}) \xleftrightarrow{\{X_l,\overline{X}_l\} \leftrightarrow \{p_l,q_l\}} E\{C_{ij}^2\}(\boldsymbol{p})(\text{s-o-p}) \qquad (3)$$

Equations (2) and (3) show that computing the mean $E\{C_{ij}\}$ and the variance of the capacity

$VAR\{C_{ij}\} = E\{C_{ij}^2\} - \left(E\{C_{ij}\}\right)^2$ can be achieved by ensuring that both the capacity itself and its square are expressed in s-o-p form.

4 A Map Procedure

A modified Karnaugh map can be used to specify the pseudo-switching (-Boolean) function $C_{ij}(\boldsymbol{X})$ [30-42]. This Karnaugh map serves as a powerful manual tool that provides pictorial insight about the various functional properties, concepts and procedures. The Karnaugh map comprises of n input variables $\boldsymbol{X}$ acting as the map variables and real numbers $C_{ij}(\boldsymbol{X}_k)$, denoting the map entries, which represent the flow capacity of the network for states $\boldsymbol{X}_k$. These real numbers can be any numerical values other than 1's and 0's. The methods demonstrated in sections 7 and 8 can be used to derive these numbers individually or collectively.

To express $C_{ij}(\boldsymbol{X})$ in its minimal sum-of-products form, it is required that entries other than zero are covered by the least number of loops on the map. The individual loop used ought to be the largest combining factor of 2^i $\{i = 0, 1, 2, \dots, n\}$ adjacent cells of the map that contains a certain minimum value (so far not covered). For such a loop, the contribution to the sum-of-products form for the pseudo switching function $C_{ij}(\boldsymbol{X})$ corresponds to the typical loop term multiplied by the value covered by the loop.

In a scenario where the selection of a larger loop is needed, entries in a cell can be divided, creating multiple values that can cover more than a few loops. Such a division is viable for entries whose values are integers that represent networks of small size. When a portion of loop entry is covered, the entry is substituted by its uncovered portion. More precisely, an entry is substituted by zero if it is covered totally. This process ends when all entries are exhausted, i.e., when replacement by zero for all entries is completed.

The process outlined above results in an expression of capacity that is less complex than that obtained by the direct state enumeration method [19]. The advantage of the map procedure is that it is efficient in cases where entries in the map comprise integral values belonging to a small set. This happens, for instance, when branch capacities include a few integer values only. The map does, however, suffer limitations in that it is capable of dealing with only small networks (usually not more than six branches). This can be remedied by extending it using variable-entered Karnaugh maps (VEKMs). That way, the map procedure can manage sizeable networks [36,38,54,58-65].

Example 1

This example applies the map procedure to the network shown in Fig. 1, whose branch capacities are: $\boldsymbol{c} = [\,6\;\;7\;\;4\;\;10\;\;5\;\;3\;\;4\,]^T$

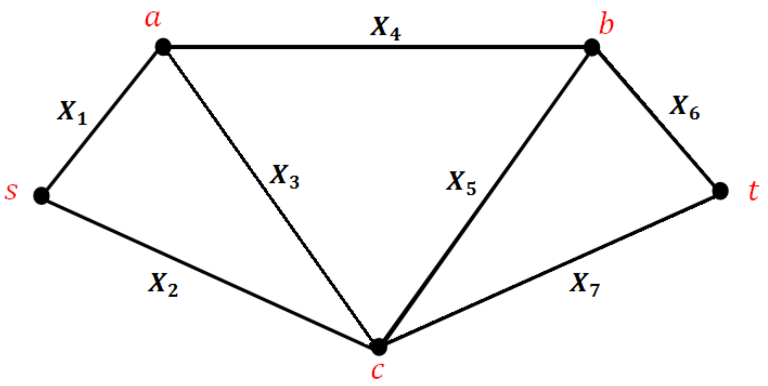
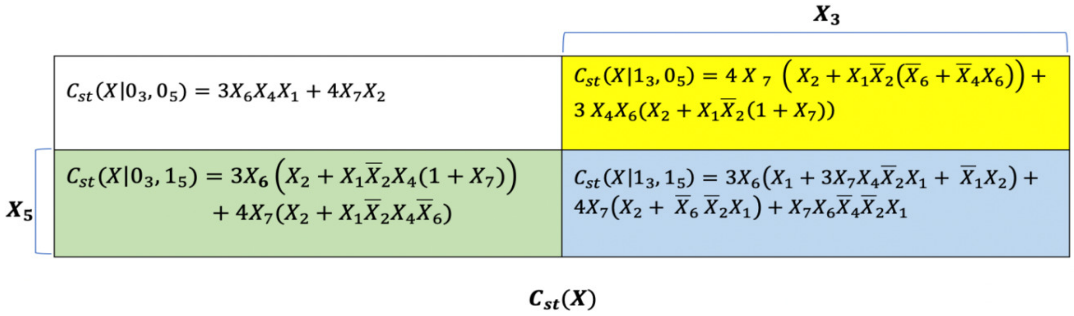
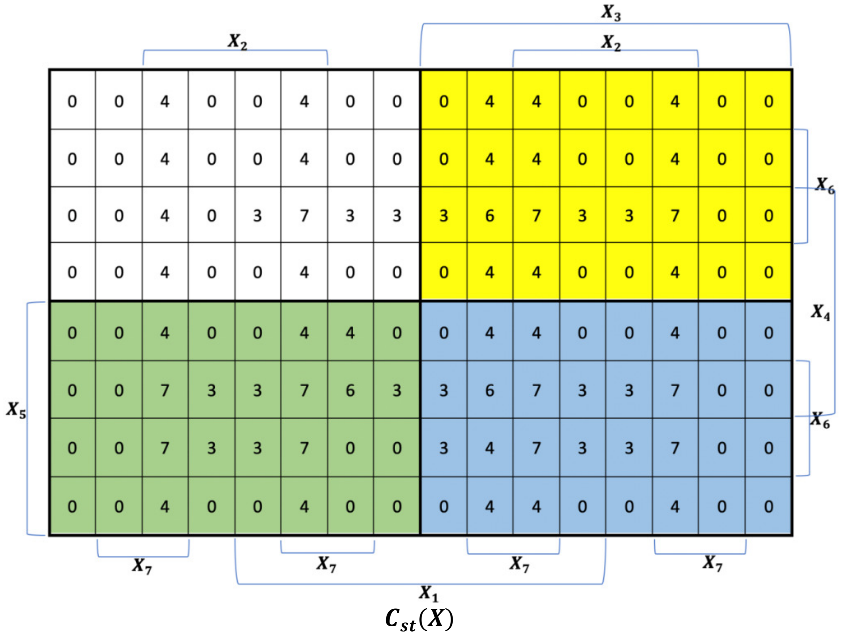


Fig. 1. A 7-branch bridge network of a capacity vector $c = [\,6\ \ 7\ \ 4\ \ 10\ \ 5\ \ 3\ \ 4\,]^T$



$C_{st}(X\|0_3,0_5) = 3X_6X_4X_1 + 4X_7X_2$	$C_{st}(X\|1_3,0_5) = 4\,X_7\left(X_2 + X_1\overline{X}_2(\overline{X}_6 + \overline{X}_4X_6)\right) +$ $3\,X_4X_6(X_2 + X_1\overline{X}_2(1 + X_7))$
$C_{st}(X\|0_3,1_5) = 3X_6\left(X_2 + X_1\overline{X}_2X_4(1 + X_7)\right)$ $+ 4X_7(X_2 + X_1\overline{X}_2X_4\overline{X}_6)$	$C_{st}(X\|1_3,1_5) = 3X_6(X_1 + 3X_7X_4\overline{X}_2X_1 + \overline{X}_1X_2) +$ $4X_7(X_2 + \overline{X}_6\,\overline{X}_2X_1) + X_7X_6\overline{X}_4\overline{X}_2X_1$

Fig. 2. A variable-entered Karnaugh map for the pseudo-Boolean function $C_{st}(X)$ with map variables $X_3 and\ X_5$ corresponding to the two bridging elements in the network of Fig. 1



0	0	4	0	0	4	0	0	0	4	4	0	0	4	0	0
0	0	4	0	0	4	0	0	0	4	4	0	0	4	0	0
0	0	4	0	3	7	3	3	3	6	7	3	3	7	0	0
0	0	4	0	0	4	0	0	0	4	4	0	0	4	0	0
0	0	4	0	0	4	4	0	0	4	4	0	0	4	0	0
0	0	7	3	3	7	6	3	3	6	7	3	3	7	0	0
0	0	7	3	3	7	0	0	3	4	7	3	3	7	0	0
0	0	4	0	0	4	0	0	0	4	4	0	0	4	0	0

Fig. 3. Karnaugh map representation of the capacity pseudo-Boolean function $C_{st}(X)$

Fig. 2 shows A variable-entered Karnaugh map for the pseudo-Boolean function $C_{st}(\boldsymbol{X})$ with map variables $X_3 \; and \; X_5$ corresponding to the two bridging elements in the network of Fig. 1. In addition, Fig. 3 shows the Karnaugh map representation of the capacity pseudo-Boolean function $C_{st}(\boldsymbol{X})$.

The map has $2^7 = 128$ cells such that each one of them depicts a definite state of the flow network. The Karnaugh map entries are the real numbers, which correspond to the integer values of the capacity pseudo-switching function $C_{st}(\boldsymbol{X})$ for states $\boldsymbol{X}$.

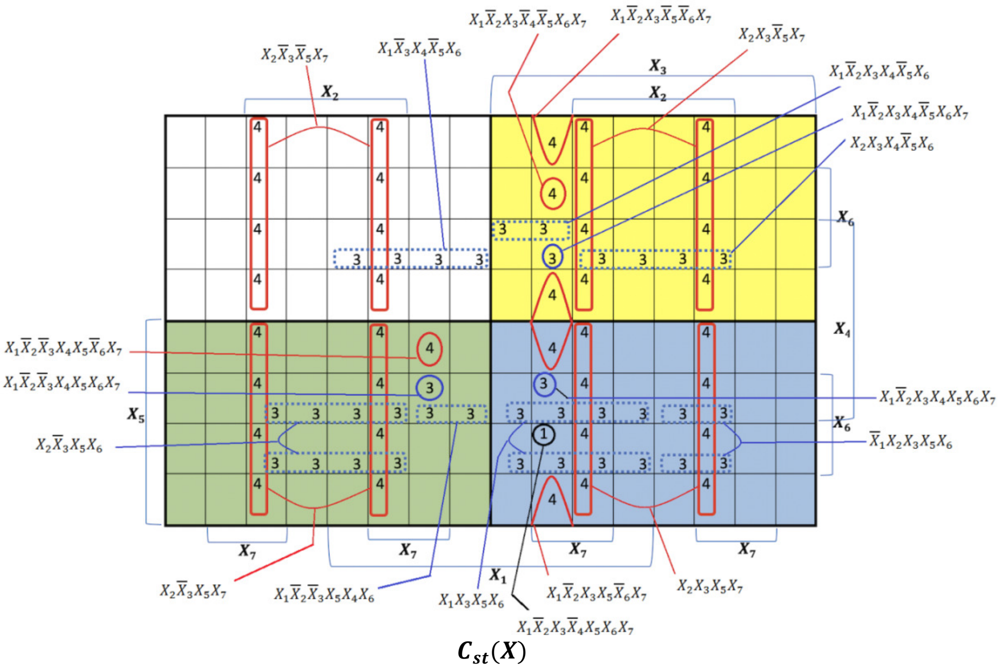


Fig. 4. A 2- Stage Map Procedure to cover the s-t capacity function $C_{st}(X)$for nonzero entries

Fig. 4 demonstrates a 2-stage map procedure to cover entries other than zero in the map representing the pseudo-switching function $C_{st}(\boldsymbol{X})$. In stage 1, this procedure covers every cell in the map that possesses an entry that is at least 4, i.e., an entry that is either 4 or 7. Therefore, the remaining entry in each of these cells will be either 0 or 3, respectively. Therefore, the task in stage 2 is to cover the map entry 3, which is the only remaining non-zero map entry. The minimal sum-of-product equation for the pseudo-switching function $C_{st}(\boldsymbol{X})$ and the corresponding one for its mean are

$$C_{st}(\boldsymbol{X}) = \overline{X}_3\overline{X}_5[\,3\,X_6X_4X_1 + 4\,X_7X_2\,] + \overline{X}_3X_5\,[3\,X_{\boldsymbol{6}}\left(X_2 + X_1\overline{X}_2X_4(1+X_7)\right) + 4\,X_7(X_2 + X_1\overline{X}_2X_4\overline{X}_6)\,] + X_3\overline{X}_5\,[\,4\,X_{\,7}\left(X_2 + X_1\overline{X}_2(\overline{X}_6 + \overline{X}_4X_6)\right) + 3\,X_4X_6(X_2 + X_1\overline{X}_2(1+X_7))] + X_3X_5\,[\,3\,X_6(X_1 + 3\,X_7X_4\overline{X}_2X_1 + \overline{X}_1X_2) + 4\,X_7(X_2 + \overline{X}_6\overline{X}_2X_1) + X_7X_6\overline{X}_4\overline{X}_2X_1\,] \tag{4}$$

$$E\{C_{st}\}(\boldsymbol{p}) = q_3q_5[3\,p_6p_4p_1 + 4\,p_7p_2] + q_3p_5[\,3\,p_6(\,p_2 + p_1q_2p_4\,(1+p_7)\,) + 4\,p_7(p_2 + p_1q_2p_4q_6)] + p_3q_5[\,4\,p_7\,(\,p_2 + p_1q_2(q_6 + q_4p_6)) + 3\,p_4p_6(p_2 + p_1q_2(1+p_7))] + p_3p_5[\,3\,p_6(p_1 + 3\,p_7p_4q_2p_1 + q_1p_2) + 4\,p_7(p_2 + q_6q_2p_1) + p_7p_6q_4q_2p_1] \tag{5}$$

The conversion from the pseudo-Boolean function $C_{st}(\boldsymbol{X})$ to the Boolean function of success $S_{st}(\boldsymbol{X})$ can be done by removing all real numbers other than unity and substituting the mathematical operators$\{+,\bullet\}$ by their logic representations, viz.,

$$\begin{aligned}&S_{st}(\boldsymbol{X}) = \overline{X}_3\overline{X}_5[\,X_6X_4X_1 \vee X_7X_2\ \,] \vee \overline{X}_3X_5\ [X_{\boldsymbol{6}}\left(X_2 \vee X_1\overline{X}_2X_4(1 \vee X_7)\right) \vee X_7(X_2 \vee X_1\overline{X}_2X_4\overline{X}_6)\,]\\ &\vee X_3\overline{X}_5\ [\,X_7\ \left(X_2 \vee X_1\overline{X}_2(\overline{X}_6 \vee \overline{X}_4X_6)\right) \vee X_4X_6(X_2 \vee X_1\overline{X}_2(1 \vee X_7))]\\ &\vee X_3X_5\ [\,X_6(X_1 \vee X_7X_4\overline{X}_2X_1 \vee \overline{X}_1X_2) \vee X_7(X_2 \vee\ \overline{X}_6\overline{X}_2X_1) \vee X_7X_6\overline{X}_4\overline{X}_2X_1\,]\end{aligned} \tag{6}$$

The final sum-of-product equation for the capacity squared $C_{st}^2(\boldsymbol{X})$ can be successfully obtained either by squaring expression (4) or by using the map technique in which all the map cell entries are squared for those of the Karnaugh map in Fig. 3. The equation for the pseudo-switching function $C_{st}^2(\boldsymbol{X})$ and its mean are

$$\begin{aligned}&C_{st}^2(\boldsymbol{X}) =\ \overline{X}_3\overline{X}_5[\,9\,X_6X_4X_1 + 16\,X_7X_2\ \,] +\ \overline{X}_3X_5\ [9\,X_{\boldsymbol{6}}\left(X_2 + X_1\overline{X}_2X_4(1 + X_7)\right) + 16\,X_7(X_2 +\\ &X_1\overline{X}_2X_4\overline{X}_6)\,] + X_3\overline{X}_5\ [\,16\,X_7\ \left(X_2 + X_1\overline{X}_2(\overline{X}_6 + \overline{X}_4X_6)\right) + 9\,X_4X_6(X_2 + X_1\overline{X}_2(1 + X_7))] +\\ &X_3X_5\ [\,9\,X_6(X_1 + 9\,X_7X_4\overline{X}_2X_1 +\ \overline{X}_1X_2) + 16\,X_7(X_2 +\ \ \overline{X}_6\overline{X}_2X_1) + X_7X_6\overline{X}_4\overline{X}_2X_1\,]\end{aligned} \tag{7}$$

$$\begin{aligned}&E\{C_{st}^2\}(\boldsymbol{p}) = q_3q_5[9\,p_6p_4p_1 + 16\,p_7p_2] + q_3p_5[\,9\,p_6(\,p_2 + p_1q_2p_4\,(1 + p_7)\,) + 16\,p_7(p_2 + p_1q_2p_4q_6)] +\\ &p_3q_5[\,16\,p_7\,(\,p_2 + p_1q_2(q_6 + q_4p_6)) + 9\,p_4p_6(p_2 + p_1q_2(1 + p_7))] + p_3p_5[\,9\,p_6(p_1 + 9\,p_7p_4q_2p_1 +\\ &q_1p_2) + 16\,p_7(p_2 + q_6q_2p_1) + p_7p_6q_4q_2p_1]\end{aligned} \tag{8}$$

5 Reduction Rules

The analysis of simple networks can be done by merging branches in series and/or in parallel using both capacity and connectivity measures. If we let l denote a single branch of capacity $C_{ij}^{l}(X_l)$, then for n series branches, the capacity function is the minimum branch capacity, namely:

$$C_{ij}(\boldsymbol{X}) = min\left\{\,C_{ij}^{(l)}(X_l)\right\} \tag{9}$$

Dually, for n parallel branches, the capacity function is the sum of branch capacities, namely:

$$C_{ij}(\boldsymbol{X}) = \sum C_{ij}^{(l)}(X_l) \tag{10}$$

The minimum and summation operators in (9) and (10) are considered over all possible values of l.The combination of parallel branches is less complex in comparison to the combination of series branches in cases where the individual branches are defined only by capacity functions depicting multivalued discrete random variables. A case like that is common, e.g., when branches are obtained as a result of preceding parallel-series reductions. Nonetheless, an individual branch capacity function is initially assumed to be a binary random variable, and hence it can be expressed as a product of the algebraic value c_l and the switching variable X_l of its successes, i.e.,

$$C_{ij}^{(l)}(X_l) = c_l\,X_l \tag{11}$$

Reduction rules for series and parallel connections reduces respectively in the latter case to

$$C_{ij}(\boldsymbol{X}) = (min\ c_l)\wedge_{l=1}^{n} X_l \tag{12}$$

$$C_{ij}(\boldsymbol{X}) =\ \textstyle\sum_{l=1}^{n} c_l\,X_l \tag{13}$$

Expression (12) shows that the ordered pair of a capacity c_l , and a success X_l describing each series connection of branches can be substituted by a single equivalent branch characterized by a binary capacity function denoted by capacity ($\min c_l$) and success ($\bigwedge_{l=1}^{n} X_l$). The above observation does not hold for expression (13), for which there are parallel connections, because the capacity function becomes non-binary. Assessment of the min function is a bit more complicated in (9) than in (12) because relation (9) only requires comparison of numerals while relation (12) requires the comparison of pseudo-Boolean (-switching) functions. However, Eq. (9) can be streamlined through the algebraic decomposition formula (1).

Example 2

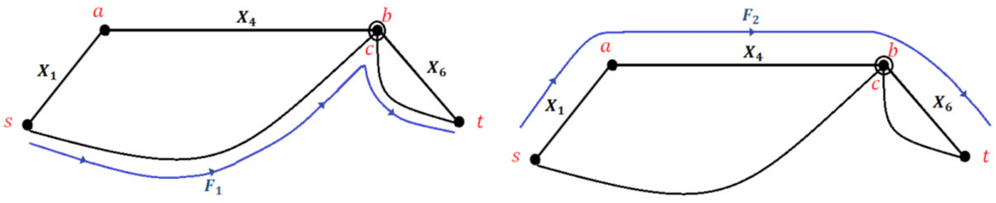
The source-to-terminal capacity function for the 7-branch bridge network in Fig. 1 can be obtained by using the algebraic decomposition rule (1). This rule is a divide-and-conquer technique that serves as a function-capacity-preserving network transformation that replaces the computation of a certain capacity function by the computation of several simpler capacity sub-functions.

The following expression which is a special case of the decomposition formula (1) is readily obtained by decomposing the s-t capacity function $C_{st}(\boldsymbol{X})$ with respect to the indicator variables X_3 and X_5 which represent the bridging components in the capacitated network of Fig. 1.

$$\begin{aligned} C_{st}(\boldsymbol{X}) = {} & \overline{X}_3\overline{X}_5\, C_{st}(X|0_3\,,0_5) + \overline{X}_3 X_5\, C_{st}(X|0_3\,,1_5) + X_3\overline{X}_5 C_{st}(X|1_3\,,0_5) \\ & + X_3X_5C_{st}\,(X|1_3\,,1_5) \end{aligned} \quad (14)$$

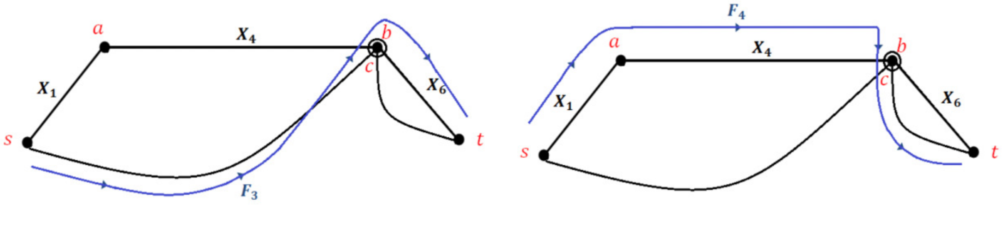
In Eq. (14), sub-function $C_{st}(\boldsymbol{X}|0_3\,,0_5)$ is the s-t capacity function of the subnetwork derived from the original flow network by opening the keystone branch 3 and 5. This subnetwork is representing a simple series-parallel system; thus, its capacity function is expressed through (12) and (13) as:

$$C_{st}(\boldsymbol{X}|0_3\,,0_5) = min\ (4,7)\ X_2X_7 + min\ (6,10,3)\ X_1X_4X_6 = 4X_2X_7 + 3X_1X_4X_6$$



Case (1): Infinite flow F_1 allowed through the originally perfect node c

Case (2): Infinite flow F_2 allowed through the originally perfect node b



Case (3): Flow F_3 is through the imperfect artificial node bc which inherits the capacity c_5 of branch X_5 , hence $F_3 \leq c_5$

Case (4): Flow F_4 is through the imperfect artificial node bc which inherits the capacity c_5 of branch X_5 , hence $F_4 \leq c_5$

Fig. 5. Pertaining to the possible failure of a subnetwork with a shorted branch to adequately represent a supposedly corresponding subnetwork $C_{st}(X|0_3\,,1_5)$

On the contrary, the other sub-functions in Eq. (14) are not representing simple series-parallel systems, though they might look like series-parallel networks at first sight. In fact, these sub-functions cannot generally be derived from the corresponding subnetworks obtained by opening an unsuccessful branch and shorting a successful one (Although these subnetworks might happen to give the correct results in the present particular case). Fig. 5 Shows four cases for the flow through perfect and imperfect nodes in the subnetwork which supposedly represents the sub-function $C_{st}(\boldsymbol{X}|0_3\,,1_5)$. Fig. 5 also illustrates the existence or non-existence of a certain limitation on the anticipated flow in each case. The reason for the afore-mentioned discrepancy is that we initially assumed nodes to be perfect, so that they are of infinite capacity and might sustain unlimited flow. Whenever, a subnetwork is created by shorting a successful branch, the two nodes at the ends of this branch are merged into a new type of node that ceases to be perfect, since it inherits the capacity of the shorted branch.

Example 3

The reduction rules for the series and parallel connections are now applied to the network of Fig. 6 which is analogous to the network in [66]. So, the network in this example can be reduced to the equivalent network in Fig. 1 via the reduction in which no further series-parallel reduction is possible.

The network has branch capacities:

$c_{1a} = 5\,,\ c_{1b} = 2\,,\ c_{1c} = 4\,,\ c_{1d} = 7\,,\ c_{1e} = 3\,,\ c_{1f} = 9\,,\ c_{2a} = 2\,,\ c_{2b} = 3\,, c_{2c} = \ 4\,, c_{3a} = 5, c_{3b} = 8\ ,$

$c_{4a} = \ 6\,, c_{4b} = \ 4\,, c_{5a} = \ 10\,, c_{5b} = \ 3\,, c_{6a} = 3\,,\ c_{6b} = 4\,,\ \ c_{6c} = 2\,, c_{6d} = 5\,, c_{7a} = 7\,,\ c_{7b} = 2,\ c_{7c} = 3\,,$

$c_{7d} = 10\,,\ c_{7e} = 6.$

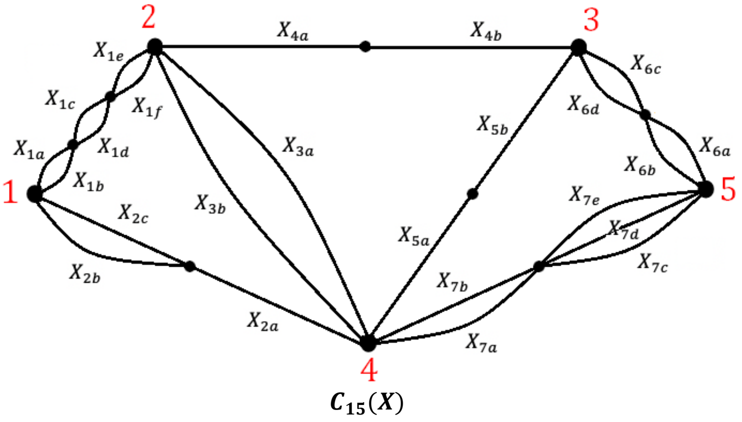


Fig. 6. This network allows further series-parallel reduction

Several capacity functions are representing a simple series-parallel system. These functions can be obtained directly through (12) and (13), e.g., $C_{24}^T(\boldsymbol{X})$, $C_{23}^T(\boldsymbol{X})$ and $C_{34}^T(\boldsymbol{X})$. However, other capacity functions are obtained via the additional use of the algebraic decomposition formula (1). These are shown by the following self- explanatory steps:

$$\begin{aligned} C_{12}^T(\boldsymbol{X}) = X_1 &= \min\{C_{1a}\,X_{1a} + C_{1b}\,X_{1b}\,, C_{1c}\,X_{1c} + C_{1d}\,X_{1d}\,, C_{1e}\,X_{1e} + C_{1f}\,X_{1f}\} \\ &= \min\{\,5\,X_{1a} + 2\,X_{1b}\,, 4\,X_{1c} + 7\,X_{1d}\,, 3\,X_{1e} + 9\,X_{1f}\} \end{aligned}$$

Decomposing the capacity function $C_{12}^{T}(\boldsymbol{X})$ with respect to the indicator variables X_{1a}, X_{1b}. The following special case of (1) is obtained:

$$\begin{aligned} C_{12}^{T}(\boldsymbol{X}) = {} & \overline{X}_{1a}\overline{X}_{1b}\, C_{12}^{T}(X|0_{1a}\,,0_{1b}) + \overline{X}_{1a}X_{1b}\, C_{12}^{T}(X|0_{1a}\,,1_{1b}) + X_{1a}\overline{X}_{1b}\, C_{12}^{T}(X|1_{1a}\,,0_{1b}) \\ & + X_{1a}X_{1b}\, C_{12}^{T}\,(X|1_{1a}\,,1_{1b}) \end{aligned}$$

$$C_{12}^{T}(\boldsymbol{X}|0_{1a}\,,0_{1b}) = 0$$

$$C_{12}^{T}(\boldsymbol{X}|0_{1a}\,,1_{1b}) = \min\{2\ ,4\,X_{1c} + 7\,X_{1d}\,,3\,X_{1e} + 9\,X_{1f}\} = 2\,(X_{1c} + X_{1d})\ (X_{1e} + X_{1f})$$

$$\begin{aligned} C_{12}^{T}(\boldsymbol{X}|1_{1a}\,,0_{1b}) &= \min\{5\ ,4\,X_{1c} + 7\,X_{1d}\,,3\,X_{1e} + 9\,X_{1f}\} \\ &= 3\,(X_{1c} + X_{1d})(X_{1e} + X_{1f}) + 2\,(X_{1d}\,X_{1f}) + X_{1c}\,\overline{X}_{1d}\,X_{1f} \end{aligned}$$

$$\begin{aligned} C_{12}^{T}\,(\boldsymbol{X}|1_{1a}\,,1_{1b}) &= \min\{7\ ,4\,X_{1c} + 7\,X_{1d}\,,3\,X_{1e} + 9\,X_{1f}\} \\ &= 3\,(X_{1c} + X_{1d})(X_{1e} + X_{1f}) + 4\,X_{1d}\,X_{1f} + X_{1c}\,\overline{X}_{1d}\,X_{1f} \end{aligned}$$

$$\begin{aligned} C_{12}^{T}(\boldsymbol{X}) = X_1 = {} & \overline{X}_{1a}X_{1b}\,\left(2\,(X_{1c} + X_{1d})\ (X_{1e} + X_{1f})\right) + X_{1a}\overline{X}_{1b}(3\,(X_{1c} + X_{1d})(X_{1e} + X_{1f}) + \\ & 2\,(X_{1d}\,X_{1f}) + X_{1c}\,\overline{X}_{1d}\,X_{1f}) + X_{1a}X_{1b}(3\,(X_{1c} + X_{1d})(X_{1e} + X_{1f}) + 4\,X_{1d}\,X_{1f} + X_{1c}\,\overline{X}_{1d}\,X_{1f}) \end{aligned} \tag{15a}$$

$$\begin{aligned} C_{14}^{T}(\boldsymbol{X}) = X_2 &= \min\{C_{2a}\,X_{2a}\,,C_{2b}\ X_{2b} + C_{2c}\,X_{2c}\} \\ &= \min\{2\,X_{2a}\,,3\,X_{2b} + 4\,X_{2c}\} = 2\,X_{2a}(X_{2b} + X_{2c}) \end{aligned} \tag{15b}$$

$$C_{24}^{T}(\boldsymbol{X}) = X_3 = C_{3a}\,X_{3a} + C_{3b}\,X_{3b} = 5\,X_{3a} + 8\,X_{3b} \tag{15c}$$

$$C_{23}^{T}(\boldsymbol{X}) = X_4 = \min(C_{4a}\,,C_{4b})\,X_{4a}\,X_{4b} = \ min\,(6\,,4)\,X_{4a}\,X_{4b} = 4\,X_{4a}\,X_{4b} \tag{15d}$$

$$C_{34}^{T}(\boldsymbol{X}) = X_5 = \min(C_{5a}\,,C_{5b})\,X_{5a}\,X_{5b} = \min(10\,,3)\,X_{5a}\,X_{5b} = 3\,X_{5a}\,X_{5b} \tag{15e}$$

$$\begin{aligned} C_{35}^{T}(\boldsymbol{X}) = X_6 &= \min\,\{C_{6a}\,X_{6a} + C_{6b}\,X_{6b}\,,C_{6c}\ X_{6c} + C_{6d}\,X_{6d}\} \\ &= \min\,\{\ 3\,X_{6a} + 4\,X_{6b}\,,2\,X_{6c} + 5\,X_{6d}\} \end{aligned} \tag{15f}$$

$$\begin{aligned} C_{45}^{T}(\boldsymbol{X}) = X_7 &= \min\{C_{7a}\,X_{7a} + C_{7b}\,X_{7b}\,,C_{7c}\,X_{7c} + C_{7d}\,X_{7d} + C_{7e}\,X_{7e}\} \\ &= \min\{7\,X_{7a} + 2\,X_{7b}\,,3\,X_{7c} + 10\,X_{7d} + 6\,X_{7e}\} \\ &= \end{aligned}$$

$$\begin{aligned} & 7\,X_{7a}X_{7d} + 2\,X_{7b}(X_{7d} + X_{7c}\overline{X}_{7d} + \ \overline{X}_{7a}\overline{X}_{7c}\overline{X}_{7d}X_{7e}) + 6\,X_{7a}\overline{X}_{7d}X_{7e} + 3\,X_{7a}\overline{X}_{7b}X_{7c}\overline{X}_{7d}\overline{X}_{7e} + \\ & X_{7a}X_{7c}\overline{X}_{7d}(X_{7b}\overline{X}_{7e} + X_{7e}) \end{aligned} \tag{15g}$$

6 General Transformation Conditions

According to the "Max-Flow Min-Cut Theorem" [12-18], the function representing the source to the terminal capacity of a network is denoted as;

$$C_{st}^{T}(\boldsymbol{X}) = \min\ \textstyle\sum_{(l,k\,)\in \mathrm{M_i}} c_{lk}X_{lk} \tag{16}$$

where M_{i} is the set of links representing the minimal source to terminal cut-set number i for the system. Hence, there are two crucial conditions for the preserving star-delta transformation in a capacitated network:

(a) There is preservation of all terminal capacity functions.
(b) There is no increment of 1-vertex cutset capacity functions [46].

Delta-star and star-delta transformations are portrayed in Fig. 7, which demonstrates both a star and a delta, with bidirectional branches [44,45]. The preservation of $C_{st}^{T}(\boldsymbol{X})$ in the transformations according to the aforementioned conditions might be reduced to the following conditions;

(a) These capacity functions $C_{12}^{T}, C_{21}^{T}, C_{13}^{T}, C_{31}^{T}, C_{23}^{T}\ and\ C_{32}^{T}$ will be preserved, viz.,

$$C_{ij}^{T} \text{ in delta} = C_{ij}^{T} \text{ in star.} \tag{17}$$

Or

$$D_{ij} + min\left(D_{ik}, D_{kj}\right) = \min(S_{in}, S_{nj}) \tag{18}$$

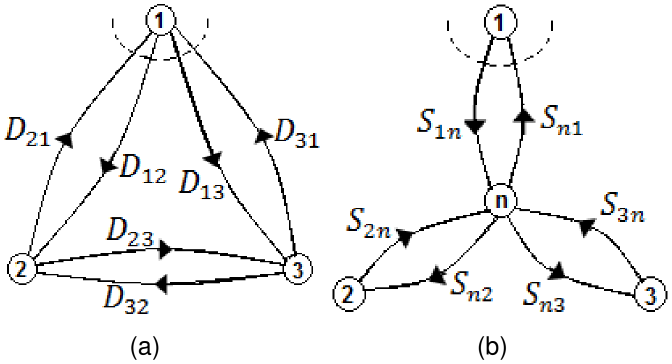


Fig. 7. (a) The delta and (b) the star, used in star-delta and delta-star transformations. The 1-vertex cut-set of node 1 is shown dotted in (a) and (b)

(b) For a delta-star transformation

$$S_{ni} \leq D_{ji} + D_{ki} \qquad S_{in} \leq D_{ij} + D_{ik} \tag{19}$$

and for a star-delta transformation

$$D_{ji} + D_{ki} \leq min\left(S_{ni}, S_{jn} + S_{kn}\right) \tag{20}$$

$$D_{ij} + D_{ik} \leq min\,(S_{in}, S_{nj} + S_{nk}) \tag{21}$$

7 Delta-Star Transformation

A delta-star transformation [44,45] presents a situation where the six D functions of the delta are known. Consequently, the six S functions of the star have to be derived via eqn. (18) subject to eqn. (19) and the non-negativity constraints $S \geq 0$. The required solutions for the six S functions are [44]

$$S_{ni} = D_{ji} + D_{ki} \qquad S_{in} = D_{ij} + D_{ik} \tag{22}$$

The preservation of the node cut-set capacity functions asserts that the terminal pair capacity functions in a delta-star transformation are preserved (i.e., eqn. (22) implies eqn. (18) in that case), and therefore (22) gives a solution that is accurate.

Given that delta branches are original, (i.e., if $D_{ij} = d_{ij}X_{ij}$), then its equivalent star is denoted by

$$S_{ni} = d_{ji}\,X_{ij} + d_{ki}\,X_{ki} \quad S_{in} = d_{ij}\,X_{ij} + d_{ik}\,X_{ki} \tag{23}$$

Equation (23) Suggests that both S_{in} and S_{ni} are pseudo-switching functions of the two indicator variables X_{ij} and X_{ki}.

Example 4

The network is shown in Fig. 8 has branch capacities $c_{12} = 6\,, c_{14} = 7\,, c_{24} = c_{42} = 4\,, c_{23} = c_{32} = 10\,, c_{34} = c_{43} = 5\,, c_{35} = 3\; and\; c_{45} = 4$. In addition to the indicator variables $X_{12} = X_1,\ X_{14} = X_2,\ X_{24} = X_{42} = X_3\,, X_{23} = X_{32} = X_4\,, X_{34} = X_{43} = X_5,\ X_{35} = X_6\,, X_{45} = X_7$.

We will apply in this network two steps of delta-star transformations. The first step is the transformation of the 2-3-4 delta to a star, which results in the equivalent star network shown in Fig. 9.

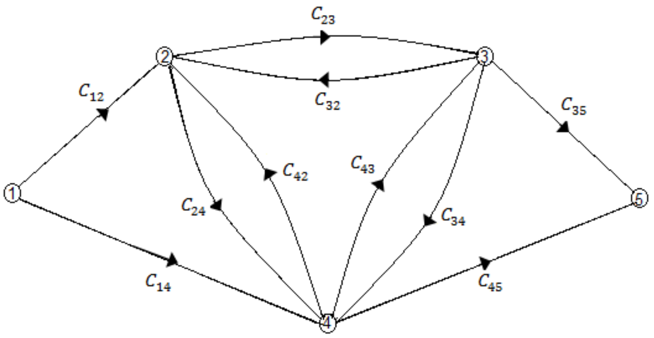


Fig. 8. A graph representing a 7-branch bridge network and a two-area power system

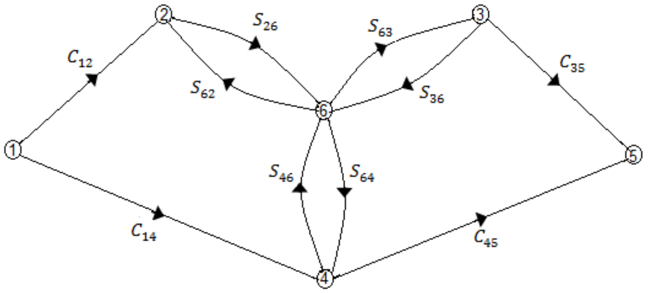


Fig. 9. A graph, equivalent to that of Fig. 8, resulting from transformation of 2-3-4 delta to star

Then, we apply a second delta-star transformation of the 4-5-6 delta in Fig. 10 after getting $C_{16}^{T}\; and\; C_{65}^{T}$ which results in the equivalent series-parallel network in Fig. 11.

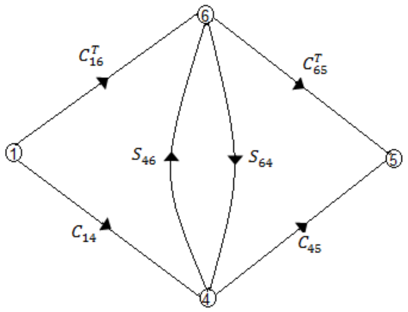


Fig. 10. Equivalent graph of Fig. 9, after getting $C_{16}^{T}\; and\; C_{65}^{T}$

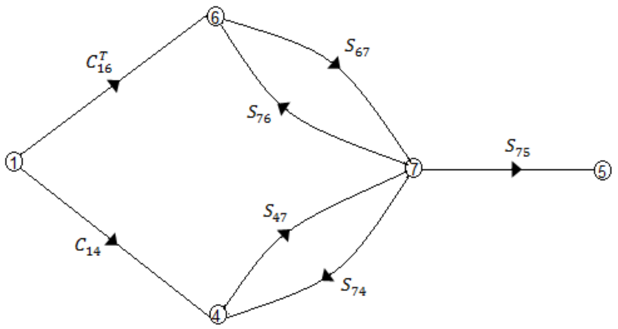


Fig. 11. A graph, equivalent to that of Fig. 10, resulting from the transformation of the 4-5-6 delta to a star

The s-t capacity function $C^T_{15}(\boldsymbol{X})$ is obtained via eqn. (1) and eqn. (23), and the series-parallel reduction rules, as shown by the following self-explanatory steps:

$$S_{26} = S_{62} = 10X_4 + 4X_3$$

$$S_{36} = S_{63} = 10X_4 + 5X_5$$

$$S_{46} = S_{64} = 5X_5 + 4X_3$$

$$\begin{aligned} C^T_{16} &= \min(6X_1, 10X_4 + 4X_3) \\ &= 4\overline{X}_4X_3X_1 + 6X_4X_1 \end{aligned}$$

$$\begin{aligned} C^T_{65} &= \min(3X_6\,, 10X_4 + 5X_5) \\ &= 3X_4X_6 + 3X_5\overline{X}_4X_6 \end{aligned}$$

$$S_{75} = C^T_{65} + 4X_7 = 3X_4X_6 + 3X_5\overline{X}_4X_6 + 4X_7$$

$$S_{67} = S_{76} = C^T_{65} + S_{46} = 3X_4X_6 + 3X_5\overline{X}_4X_6 + 5X_5 + 4X_3$$

$$S_{47} = S_{74} = 4X_7 + S_{46} = 4X_7 + 5X_5 + 4X_3$$

$$\begin{aligned} C^2_{17} &= \min(C^T_{16}\,, S_{67}) \\ &\min\left(4\overline{X}_4X_3\mathrm{X}_1 + 6\mathrm{X}_4\mathrm{X}_1\,, 3\mathrm{X}_4\mathrm{X}_6 + 3\mathrm{X}_5\overline{\mathrm{X}}_4\mathrm{X}_6 + 5\mathrm{X}_5 + 4\mathrm{X}_3\right) \\ &= 6X_6X_1X_3X_4 + 4\overline{X}_6X_1X_3X_4 + 2X_5\overline{X}_6X_1X_3X_4 + 3X_6X_1\overline{X}_3X_4 + 3X_5X_6X_1\overline{X}_3X_4 \\ &\quad +5X_5\overline{X}_6X_1\overline{X}_3X_4 + 4X_6X_1X_3\overline{X}_4 + 4\overline{X}_6X_1X_3\overline{X}_4 \end{aligned}$$

$$\begin{aligned} C^3_{17} &= \min(7X_2\,, S_{47}) = \min(7\mathrm{X}_2\,, 4X_7 + 5X_5 + 4X_3) \\ &= 7X_5X_7X_2 + 4\overline{X}_5X_7X_2 + 3X_3\overline{X}_5X_7X_2 + 5X_5\overline{X}_7X_2 + 2X_3X_5\overline{X}_7X_2 + 4X_3\overline{X}_5\overline{X}_7X_2 \end{aligned}$$

$$\begin{aligned} C^T_{17} &= C^2_{17} + C^3_{17} \\ &= 6X_6X_1X_3X_4 + 4\overline{X}_6X_1X_3X_4 + 2X_5\overline{X}_6X_1X_3X_4 + 3X_6X_1\overline{X}_3X_4 + 3X_5X_6X_1\overline{X}_3X_4 \\ &\quad +5X_5\overline{X}_6X_1\overline{X}_3X_4 + 4X_6X_1X_3\overline{X}_4 + 4\overline{X}_6X_1X_3\overline{X}_4 + 7X_5X_7X_2 + 4\overline{X}_5X_7X_2 + 3X_3\overline{X}_5X_7X_2 \\ &\quad +5X_5\overline{X}_7X_2 + 2X_3X_5\overline{X}_7X_2 + 4X_3\overline{X}_5\overline{X}_7X_2 \end{aligned}$$

$$\begin{aligned} C^T_{15} &= \min\,(C^T_{17}\,, S_{75}) \\ &= \min(6X_6X_1X_3X_4 + 4\overline{X}_6X_1X_3X_4 + 2X_5\overline{X}_6X_1X_3X_4 + 3X_6X_1\overline{X}_3X_4 + 3X_5X_6X_1\overline{X}_3X_4 \\ &\quad +5X_5\overline{X}_6X_1\overline{X}_3X_4 + 4X_6X_1X_3\overline{X}_4 + 4\overline{X}_6X_1X_3\overline{X}_4 + 7X_5X_7X_2 + 4\overline{X}_5X_7X_2 + 3X_3\overline{X}_5X_7X_2 \\ &\quad +5X_5\overline{X}_7X_2 + 2X_3X_5\overline{X}_7X_2 + 4X_3\overline{X}_5\overline{X}_7X_2\,, 3X_4X_6 + 3X_5\overline{X}_4X_6 + 4X_7) \end{aligned}$$

The decomposition relation (1) is used to obtain the general capacity function C_{15}^{T} for the bridge network $(X_3\,, X_5)$. So, Decomposing the capacity function C_{15}^{T} with respect to the indicator variables X_3 , and X_5 that represent the bridging elements in the network of Fig. 8, we obtain the following special case of (1) (which replicates (14) and repeated for convenience)

$$\begin{aligned} C_{15}^{T}(\boldsymbol{X}) = {} & \overline{X}_3\overline{X}_5\, C_{15}^{T}(X|0_3\,,0_5) + \overline{X}_3 X_5\, C_{15}^{T}(X|0_3\,,1_5) + X_3\overline{X}_5\, C_{15}^{T}(X|1_3\,,0_5) \\ & + X_3 X_5\, C_{15}^{T}\,(X|1_3\,,1_5) \end{aligned}$$

$$C_{15}^{T}(\boldsymbol{X}|0_3,0_5) = \min\,(3X_6X_1X_4 + 4X_7X_2\,, 3X_6X_4 + 4X_7)$$

$$= X_4 \min(3X_6X_1 + 4X_7X_2\,, 3X_6 + 4X_7) + \overline{X}_4 \min\,(4X_7X_2\,, 4X_7)$$

$$\begin{aligned} & = X_4\{X_1 \min(\,3X_6 + 4X_7X_2\,,\; 3X_6 + 4X_7) + \overline{X}_1 \min\,(\,4X_7X_2, 3X_6 + 4X_7)\} \\ & \quad + \overline{X}_4\{X_1 \min(4X_7X_2, 4X_7) + \overline{X}_1 \min(4X_7X_2, 4X_7)\} \end{aligned}$$

$$\begin{aligned} & = X_4\big\{X_1[\,X_6 \min(3 + 4X_7X_2\,, 3 + 4X_7) + \overline{X}_6 \min(4\mathrm{X}_7\mathrm{X}_2\,, 4\mathrm{X}_7)\big] \\ & \quad + \overline{\mathrm{X}}_1[\mathrm{X}_6 \min(4\mathrm{X}_7\mathrm{X}_2\,, 3 + 4\mathrm{X}_7) + \overline{\mathrm{X}}_6\; \min(4X_7\mathrm{X}_2\,, 4X_7)]\} \\ & \quad + \overline{X}_4\{X_1[X_6 \min(4X_7X_2\,, 4X_7)\,\overline{X}_6 \min(4X_7X_2\,, 4X_7)] \\ & \quad + \overline{X}_1[X_6 \min(4X_7X_2\,, 4X_7) + \overline{X}_6 \min(4X_7X_2\,, 4X_7)]\} \end{aligned}$$

$$\begin{aligned} & = X_4\big\{X_1\big\{X_6[X_2 \min(3 + 4X_7\;\,, 3 + 4X_7) + \overline{X}_2 \min(3\,, 3 + 4\mathrm{X}_7)\,] + \overline{X}_6[X_2 \min(4X_7, 4X_7)]\big\} + \\ & \overline{X}_1\big\{X_6[X_2 \min(4X_7, 3 + 4X_7)] \;+\; \overline{X}_6[X_2 \min(4X_7\,, 4X_7)]\big\}\,\big\} \end{aligned}$$

$$\begin{aligned} & + \overline{X}_4\Big\{\,X_1\big\{X_6[X_2\, min(\,4X_7, 4X_7)] + \overline{X}_6\;[X_2\, min(4X_7, 4X_7)]\big\} + \\ & \qquad \overline{X}_1\big\{X_6[X_2\, min(\,4X_7\,, 4X_7)] + \overline{X}_6\;[X_2\, min(4X_7, 4X_7)]\big\}\Big\} \end{aligned}$$

$$\begin{aligned} = {} & X_4\;\Big\{\;X_1\big\{X_6\big[X_2(3 + 4X_7) + \overline{X}_2(3)\big] + \overline{X}_6[X_2(4X_7)]\big\} \;+\; \overline{X}_1\big\{X_6[\,X_2(4X_7)] + \overline{X}_6[X_2(4X_7)]\big]\big\}\Big\} \\ & + \overline{X}_4\Big\{\,X_1\big\{X_6[X_2(4X_7)] + \overline{X}_6[X_2(4X_7)]\big\} + \overline{X}_1\big\{X_6[X_2(4X_7)] + \;\overline{X}_6[X_2(4X_7)\,]\big\}\Big\} \end{aligned}$$

$$= 3X_6X_4X_1 + 4X_7X_2$$

$$C_{15}^{T}(\boldsymbol{X}|1_3,1_5) = 3X_6\big(X_1 + 3X_7X_4\overline{X}_2X_1 + \overline{X}_1X_2\big) + 4X_7\big(X_2 + \overline{X}_6\,\overline{X}_2X_1\big) + X_7X_6\overline{X}_4\overline{X}_2X_1$$

$$C_{15}^{T}(\boldsymbol{X}|0_3,1_5) = 3X_{\boldsymbol{6}}\,\Big(X_2 + X_1\overline{X}_2X_4(1 + X_7)\Big) + 4X_7(X_2 + X_1\overline{X}_2X_4\overline{X}_6)$$

$$C_{15}^{T}(\boldsymbol{X}|1_3,0_5) = 4X_7\;\big(\,X_2 + X_1\overline{X}_2(\overline{X}_6 + \overline{X}_4X_6)\big) + 3X_4X_6(X_2 + X_1\overline{X}_2(1 + X_7))$$

$$\begin{aligned} C_{15}^{T}(\boldsymbol{X}) = {} & \overline{X}_3\overline{X}_5[\,3X_6X_4X_1 + 4X_7X_2\;\,] + \overline{X}_3X_5\;[3X_{\boldsymbol{6}}\,\Big(X_2 + X_1\overline{X}_2X_4(1 + X_7)\Big) + 4X_7(X_2 \\ & + X_1\overline{X}_2X_4\overline{X}_6)\,] + X_3\overline{X}_5\;[\,4X_7\;\Big(\,X_2 + X_1\overline{X}_2\big(\overline{X}_6 + \overline{X}_4X_6\big)\Big) + 3X_4X_6(X_2 \end{aligned} \tag{24}$$

$$+X_1\overline{X}_2(1 + X_7))] \;+ X_3X_5\;[\,3X_6\big(X_1 + 3X_7X_4\overline{X}_2X_1 + \overline{X}_1X_2\big) + 4X_7\big(X_2 + \overline{X}_6\overline{X}_2X_1\big) + X_7X_6\overline{X}_4\overline{X}_2X_1\;]$$

Expression (24) for $C_{15}^{T}(X)$ is equivalent to equation (4) and equation (33) in the generalized cutset procedure.

8 A Generalized Cutset Procedure

Calculating the maximum flow is one of the most critical problems in a capacitated network. The complex part in this calculation is to ensure that the branch capacity is not violated when determining the maximum number of flow units from the source node to the terminal node. An inherent postulation in that issue is that not a single network branch is failed, i.e., $\boldsymbol{X} = 1$ represents the network state, and $C_{st}(\mathbf{1})$ represents the maximum flow. The maximum flow algorithm of Ford & Fulkerson can thus be introduced to compute the maximum flow problem. The upshot of the application is the celebrated "Max-Flow Min-Cut Theorem," This approach can be put in a general form that accounts for all network states as follows

$$C_{st}(\boldsymbol{X}) = \min\{ \textstyle\sum_{l \in M_i} c_l X_l \}, \tag{25}$$

where M_{i} is the set of links representing the minimal source to terminal cut-set number i for the system [46]. Equation (25) includes as special cases the series-parallel reduction rules, which were shown in the previous section.

To apply (25) in the calculation of $C_{st}(\boldsymbol{X})$, we note that $C_{st}(\boldsymbol{X}) = 0$ for all cases where the state $\boldsymbol{X}$ is an s-t cutset. This holds if a connection between s and t does not exist. On the other hand, $C_{st}(\boldsymbol{X}) \neq 0$ whenever state $\boldsymbol{X}$ denotes a path between s and t. Thus, we let $\{P_j\}$ denote a (preferably minimal) set of s-t paths that are exhaustive and disjoint [30], i.e., if

$$S_{st} = \textstyle\bigvee_{j=1}^{n_p} P_j\,, \tag{26}$$

$$P_j \wedge P_k = 0\,, for\ all\ j \neq k\,, \tag{27}$$

where S_{st} is the indicator variable for successful operation of the flow network which can be equivalent to connectivity [46,47,52,54,55], then $C_{st}(\boldsymbol{X})$ is:

$$C_{st}(\boldsymbol{X}) = \textstyle\sum_{J=1}^{n_p} P_j\, C_{st}(X\,|P_j = 1). \tag{28}$$

Repeated application of the algebraic decomposition formula (1) can be used to prove equation (28). Through substituting $\{\boldsymbol{X} \mid P_j = 1\}$ for $\boldsymbol{X}$ in (25), the sub-function $C_{st}(\boldsymbol{X}\,|P_j = 1)$ in (28) can be derived.

Example 5

The problem of Examples 2 and 4 is now revisited by applying the "Max-Flow Min-Cut Theorem". The capacitated network of Fig. 1 has 6 minimal cut-sets, whose capacities are given by

$C_1 = \{6X_1 + 7X_2\}$, $C_2 = \{3X_6 + 4X_7\}$, $C_3 = \{10X_4 + 4X_3 + 7X_2\}$, $C_4 = \{10X_4 + 5X_5 + 4X_7\}$,
$C_5 = \{6X_1 + 4X_3 + 5X_5 + 4X_7\}$ and $C_6 = \{7X_2 + 4X_3 + 5X_5 + 3X_6\}$

Thus, expression (25) takes the form:

$$C_{st}(\boldsymbol{X}) = \min\,(c_1X_1 + c_2X_2\,,\ c_6X_6 + c_7X_7\,, c_4X_4 + c_3X_3 + c_2X_2\,,\ c_4X_4 + c_5X_5 + c_7X_7\,, c_1X_1 + c_3X_3 + c_5X_5 + c_7X_7\,,\ c_2X_2 + c_3X_3 + c_5X_5 + c_6X_6\,)$$

$$= \min(6X_1 + 7X_2\,,\ 3X_6 + 4X_7\,, 10X_4 + 4X_3 + 7X_2\,,\ 10X_4 + 5X_5 + 4X_7\,, 6X_1 + 4X_3 + 5X_5 + 4X_7\,,\ 7X_2 + 4X_3 + 5X_5 + 3X_6\,) \tag{29}$$

Expression (29) can be streamlined through the algebraic decomposition rule (1). We now employ (14) for decomposing the capacity function $C_{st}(\boldsymbol{X})$ with respect to the indicator variables (X_3, X_5) that represent the bridge elements in the network of Fig. 1. The sub-functions in (14) are obtained via (29) as:

$$C_{st}(\boldsymbol{X}|1_3, 1_5) = \ min\ (\ 6X_1 + 7X_2\ ,\ 3X_6 + 4X_7\ , 10X_4 + 4 + 7X_2\ ,\ 10X_4 + 5 + 4X_7\ , 6X_1 +\ 9 + 4X_7\ ,\ 7X_2 + 9\ + 3X_6\ )$$

$$\begin{aligned} &= X_1\, min(\, 6 + 7\, X_2\, , 3\, X_6 + 4\, X_7\, , 10\, X_4 + 4 + 7\, X_2\, , 10\, X_4 + 5 + 4\, X_7\, , 15 + 4\, X_7\, , 7\, X_2 + \\ &\quad 9 + 3\, X_6)\ +\ \overline{X}_1\, min(\, 7\, X_2\, , 3\, X_6 + 4\, X_7\, , 10\, X_4 + 4 + 7\, X_2\, , 10\, X_4 + 5 + 4\, X_7, 9 + \\ &\quad 4\, X_7\, , 7\, X_2 + 9 + 3\, X_6) \end{aligned}$$

$$\begin{aligned} &= X_1\, \{X_2\, min(\, 13\, , 3\, X_6 + 4\, X_7\, , 10\, X_4 + 11\, , 10\, X_4 + 5 + 4\, X_7, 15 + 4\, X_7, 16 + 3\, X_6)\ + \\ &\overline{X}_2\, min(\, 6\, , 3\, X_6 + 4\, X_7, 10\, X_4 + 4\, , 10\, X_4 + 5 + 4\, X_7\, , 15 + 4\, X_7, 9 + 3\, X_6)\} + \\ &\overline{X}_1\, \{\, X_2 min(\, 7\, , 3\, X_6 + 4\, X_7, 10\, X_4 + 11\, , 10\, X_4 + 5 + 4\, X_7, 9 + 4\, X_7, 16 + 3\, X_6)\} \end{aligned}$$

$$\begin{aligned} = \\ X_1\, \{\, X_2\, \{X_4\, min(\, 13\, , 3\, X_6 + 4\, X_7\, , 21\, , 15 + 4\, X_7\, , 15 + 4\, X_7\, , 16 + 3\, X_6) + \\ &\overline{X}_4\, min(\, 13\, , 3\, X_6 + 4\, X_7\, , 11\, , 5 + 4\, X_7\, , 15 + 4\, X_7, 16 + 3\, X_6)\} + \\ &\overline{X}_2\, \{X_4\, min(\, 6\, , 3\, X_6 + 4\, X_7, 14\ \, , 15 + 4\, X_7\, , 15 + 4\, X_7\, , 9 + 3\, X_6\,) + \\ &\overline{X}_4\, min(\, 6\, , 3\, X_6 + 4\, X_7, 4\, , 5 + 4\, X_7\, , 15 + 4\, X_7\, , 9 + 3\, X_6\,)\, \}\, \} + \\ &\overline{X}_1\ \left\{ X_2\, \{\, X_4\, min(\, 7\, , 3\, X_6 + 4\, X_7, 21\, , 15 + 4\, X_7, 9 + 4\, X_7, 16 + 3\, X_6) + \right. \\ &\left. \overline{X}_4\, min(\, 7\, , 3\, X_6 + 4\, X_7\, , 11\, , 5 + 4\, X_7\, , 9 + 4\, X_7\, , 16 + 3\, X_6\,)\}\right\} \end{aligned}$$

$$\begin{aligned} &= X_1\, \{\, X\ _2\ \ \{\, X_4\, [\, X_6\, (\, 3 + 4\, X_7\,) + \ \overline{X}_6\, (4\, X_7)\,] + \ \overline{X}_4\, [\, X_6\, (\, 3 + 4\, X_7\,)\ +\ \ \overline{X}_6(4\, X_7)]\ \} + \\ &\quad \overline{X}_2\, \{\, X_4\, [\, X_6(\, 3 + 3\, X_7\,) + \ \overline{X}_6(\, 4\, X_7)]\ +\ \overline{X}_4\, [\, X_6\, (\, 3 + X_7\,) + \ \ \overline{X}_6(\, 4\, X_7\,)]\, \}\ \} \\ &\quad + \overline{X}_1\ \{\, X_2\, \{\, X_4\, [\, X_6(\, 3 + 4\, X_7\,) + \ \overline{X}_6(\, 4\, X_7)\,] + \ \overline{X}_4[\, X_6\, (\, 3 +\ \ 4\, X_7\,) + \ \overline{X}_6\, (\, 4\, X_7)\,]\, \}\, \} \end{aligned}$$

$$= \ 3X_6\big(X_1 + X_7X_4\overline{X}_2X_1 + \ \overline{X}_1X_2\big) + 4X_7\big(X_2 + \ \overline{X}_6\, \overline{X}_2X_1\big) + X_7X_6\overline{X}_4\overline{X}_2X_1 \qquad (30a)$$

$$C_{st}(\boldsymbol{X}|0_3, 1_5) = 3X_{\boldsymbol{6}}\left(X_2 + X_1\overline{X}_2X_4(1 + X_7)\right) + 4X_7(X_2 + X_1\overline{X}_2X_4\overline{X}_6) \qquad (30b)$$

$$C_{st}(\boldsymbol{X}|1_3, 0_5) = 4X_7\ \big(\, X_2 + X_1\overline{X}_2(\overline{X}_6 + \overline{X}_4X_6)\big) + 3X_4X_6(X_2 + X_1\overline{X}_2(1 + X_7)) \qquad (30c)$$

$$C_{st}(\boldsymbol{X}|0_3, 0_5) = 3X_6X_4X_1 + 4X_7X_2 \qquad (30d)$$

These sub-functions can be substituted into (14) to get the equivalent form (4). They can also be used to fill in the map entries in Fig. 2 and Fig. 3.

On the other hand, (29) can be successfully simplified by using eq. (28). A set of exhaustive and disjoint s-t paths for the network is:

$$\begin{aligned} &P_1 = X_2X_7\ ,\ \ P_2 = X_2X_3X_4\overline{X}_5X_6\overline{X}_7\ ,\ P_3 = X_2X_5X_6\overline{X}_7\ ,\ P_4 = X_1X_2\overline{X}_3X_4\overline{X}_5X_6\overline{X}_7\ \ , \\ &P_5 = X_1\overline{X}_2\overline{X}_3X_4X_6\ ,\ \ P_6 = X_1\overline{X}_2\overline{X}_3X_4X_5\overline{X}_6X_7\ ,\ \ P_7 = X_1\overline{X}_2X_3X_5X_6\overline{X}_7\ , \\ &\ \ P_8 = X_1\overline{X}_2X_3X_7\ \ ,\ \ P_9 = X_1\ \overline{X}_2X_3X_4\overline{X}_5X_6\overline{X}_7 \end{aligned}$$

Therefore, the sub-functions in (28) are obtained via (29) as

$$\begin{aligned} C_{st}\ (\, \boldsymbol{X}\ |P_1 = 1\,) &= C_{st}\ (\, X\ |1_2\, , 1_7\,) \\ &= min\ (\, 6X_1 + 7\, ,\ 3X_6 + 4\, , 10X_4 + 4X_3 + 7\ \ ,\ 10X_4 + 5X_5 + 4\, , 6X_1 + 4X_3 + 5X_5 + 4\ \ , \\ &\quad 7 + 4X_3 + 5X_5 + 3X_6\,) \\ &= \ 4 + 3\, X_6\ \left(\, X_5 + X_4\, \overline{X}_5\ (\, 1 + X_1\overline{X}_3\,)\right) \qquad (31a) \end{aligned}$$

$$\begin{aligned} C_{st}(\, \boldsymbol{X}\ |P_2 = 1\,) &= C_{st}(\, X\ |1_2, 1_3, 1_4, 0_5, 1_6, 0_7\,) \\ &= min(\, 6\, X_1 + 7\ \ , 3\, , 21\, , 10\, , 6\, X_1 + 4\, , 14\ \ ) = \ 3 \qquad (31b) \end{aligned}$$

$$C_{st}\ (\boldsymbol{X} \mid P_3 = 1) = C_{st}\ (X\ | 1_2\ , 1_5\ , 1_6\ , 0_7\ ) \\ = min(\ 6\ X_1 + 7\ \ , 3\ ,\ \ 10\ X_4 + 4\ X_3 + 7\ , 10\ X_4 + 5\ , 6\ X_1 + 4\ X_3\ +\ \ \ 5\ , 7 + 4\ X_3 + 8\ ) = 3 \quad (31c)$$

$$C_{st}\ (\boldsymbol{X}|P_4 = 1\ ) = C_{st}(\ X\ | 1_1\ , 1_2\ , 0_3\ , 1_4\ , 0_5\ , 1_6\ , 0_7\ )\ \ =\ \ 3 \quad (31d)$$

$$C_{st}(\ \boldsymbol{X} \mid P_5 = 1\ ) = C_{st}(\ X\ | 1_1\ , 0_2\ , 0_3\ , 1_4\ , 1_6)\ \ =\ \ 3 + 3\ X_5X_7 \quad (31e)$$

$$C_{st}(\ \boldsymbol{X} \mid P_6 = 1\ ) = C_{st}(\ X \mid 1_1, 0_2, 0_3, 1_4, 1_5, 0_6, 1_7)\ \ =\ \ 4 \quad (31f)$$

$$C_{st}(\ \boldsymbol{X} \mid P_7 = 1\ ) = C_{st}(\ X \mid 1_1, 0_2, 1_3, 1_5, 1_6, 0_7)\ \ =\ \ 3 \quad (31g)$$

$$C_{st}(\ \boldsymbol{X} \mid P_8 = 1\ ) = C_{st}(\ X \mid 1_1, 0_2, 1_3, 1_7) \\ = min\ (\ 6\ ,\ \ 3X_6 + 4\ , 10X_4 + 4\ \ ,\ \ 10X_4 + 5X_5 +\ \ 4\ , 14 +\ \ \ 5X_5\ ,\ \ 4 + 5X_5 + 3X_6\ ) \\ =\ \ 4 + 2X_4X_6 \quad (31h)$$

$$C_{st}(\ \boldsymbol{X} \mid P_9 = 1\ ) = C_{st}(\ X \mid 1_1, 0_2, 1_3, 1_4, 0_5, 1_6, 0_7) =\ \ 3 \quad (31i)$$

These sub-functions, can together with the condition $C_{st}(\boldsymbol{X}\ |S_{st} = 0) = 0,$ be used to fill in the map entries in Fig. 2 and Fig. 3. Moreover, they can be substituted into (28) to get the expression:

$$C_{st}(\boldsymbol{X}) = X_2X_7\ \left(4 + 3\ X_6\ \left(X_5 + X_4\ \overline{X}_5\ \left(\ 1 + X_1\overline{X}_3\ \right)\right)\right) + 3\ X_2X_3X_4\overline{X}_5X_6\overline{X}_7 + 3\ X_2X_5X_6\overline{X}_7 + \\ 3\ X_1X_2\overline{X}_3X_4\overline{X}_5X_6\overline{X}_7 + X_1\overline{X}_2\overline{X}_3X_4X_6(\ 3 + 3\ X_5X_7\ ) + 4\ X_1\overline{X}_2\overline{X}_3X_4X_5\overline{X}_6X_7 + 3\ X_1\overline{X}_2X_3X_5X_6\overline{X}_7 + \\ X_1\overline{X}_2X_3X_7(\ 4 + 2\ X_4X_6) + 3\ X_1\overline{X}_2X_3X_4\overline{X}_5X_6\overline{X}_7 \quad (32)$$

Expression (32) can be shown to be an equivalent to expression (4) obtained by the map procedure and expression (24) obtained by the delta-star transformation procedure.

9 Conclusions

This paper presents several kinds of methods for analyzing a capacitated flow network such as telecommunication networks, power transmission systems or oil or water pipeline systems. These methods include Karnaugh maps, network reduction rules associated with delta-star transformations that preserve the source to terminal capacity function, and a generalization of the min-max cut theorem. The network capacity is a pseudo-Boolean function of the link successes; thus, its mean value is easily obtainable from its sum-of-products expression. To demonstrate the presented methods applicability, five demonstrative inter-related examples are given with sufficiently explanatory details.

Competing Interests

Authors have declared that no competing interests exist.

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