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A Study on the Sums of Squares of Generalized Fibonacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kx^k W_k^2$ and $\sum_{k=1}^{n} kx^k W_{-k}^2$

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kx^k W_k^2$ and $\sum_{k=1}^{n} kx^k W_{-k}^2$ for the squares of generalized Fibonacci numbers are presented. As special cases, we give sum formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery. Our work generalize second order recurrence relations.

Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Jacobsthal numbers; sum formulas; summing formulas.

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1 INTRODUCTION

The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \;\; n \geq 2, \quad F_0 = 0, \; F_1 = 1.$$

The generalization of Fibonacci sequence leads to several nice and interesting sequences. Horadam [1] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \;\; W_0 = c, \; W_1 = d, \quad (n \geq 2) \quad (1.1)$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [2], [3] and [4]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of c, d, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences

Name of sequence	Notation: $W_n(c, d; r, s)$	OEIS: [5]
Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=0}^{n} k(-1)^k F_k^2 = \frac{1}{25}((-1)^n\,((5n^2+10n-31)F_{n+2}^2+(-5n^2+5n+41)F_{n+1}^2-(5n^2+15n-26)F_{n+2}F_{n+1})-36)$$

and

$$\sum_{k=1}^{n} k(-1)^k P_{-k}^2 = \frac{1}{32}((-1)^n\,((-4n^2+27)P_{-n+1}^2+(4n^2+24n-23)P_{-n}^2+(8n^2-8n-54)P_{-n+1}P_{-n})-27).$$

In this work, we derive expressions for sums of second powers of generalized Fibonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[6,7,8,9,10,11,12]	[13,14,15,16,17,18,19]	[20,21,22]
Generalized Tribonacci	[23,24,25]		
Generalized Tetranacci	[26,27,28]		

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 1.1. *Let x be a complex number. If $(sx + 1)\,(r^2x - s^2x^2 + 2sx - 1) \neq 0$ then*

(a)

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta_1}{(sx+1)(r^2x - s^2x^2 + 2sx - 1)}.$$

where

$$\begin{aligned}\Delta_1 &= (-(sx-1)x^{n+2}W_{n+2}^2 - (r^2x + sx + r^2sx^2 - 1)x^{n+1}W_{n+1}^2 \\ &\quad +2rsx^{n+3}W_{n+2}W_{n+1} + x(sx-1)W_1^2 + (r^2x + sx + r^2sx^2 - 1)W_0^2 - 2rsx^2W_1W_0).\end{aligned}$$

(b)

$$\sum_{k=0}^{n} x^k W_{k+1}W_k = \frac{\Delta_2}{(sx+1)(r^2x - s^2x^2 + 2sx - 1)}.$$

where

$$\begin{aligned}\Delta_2 &= (rx^{n+2}W_{n+2}^2 + rs^2x^{n+3}W_{n+1}^2 - (r^2x + s^2x^2 - 1)x^{n+1}W_{n+2}W_{n+1} \\ &\quad -rxW_1^2 - rs^2x^2W_0^2 + (r^2x + s^2x^2 - 1)W_1W_0).\end{aligned}$$

Proof. This is given in [12].

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 1.2. *Let x be a complex number. If $(s+x)(r^2x + 2sx - s^2 - x^2) \neq 0$ then*

(a)

$$\sum_{k=1}^{n} x^k W_{-k}^2 = \frac{\Delta_3}{(s+x)(r^2x + 2sx - s^2 - x^2)}$$

where

$$\begin{aligned}\Delta_3 &= x^{n+1}(s-x)W_{-n+1}^2 + x^{n+1}(r^2s + r^2x + sx - x^2)W_{-n}^2 - 2rsx^{n+1}W_{-n+1}W_{-n} \\ &\quad +x(x-s)W_1^2 + x(-r^2s - r^2x - sx + x^2)W_0^2 + 2rsxW_1W_0.\end{aligned}$$

(b)

$$\sum_{k=1}^{n} x^k W_{-k+1}W_{-k} = \frac{\Delta_4}{(s+x)(r^2x + 2sx - s^2 - x^2)}$$

where

$$\begin{aligned}\Delta_4 &= (-rx^{n+2}W_{-n+1}^2 - rs^2x^{n+1}W_{-n}^2 + x^{n+1}(r^2x + s^2 - x^2)W_{-n+1}W_{-n} \\ &\quad +rx^2W_1^2 + rs^2xW_0^2 - x(r^2x + s^2 - x^2)W_1W_0).\end{aligned}$$

Proof. This is given in [12].

2 AN APPLICATION OF THE SUM OF THE SQUARES OF THE NUMBERS

An application of the sum of the squares of the numbers is circulant matrix. Circulant matrices have been extensively used in many scientific areas such as image processing, coding theory and signal processing, numerical analysis, optimization, mathematical statistics and modern technology.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. For generalized m-step Fibonacci sequences see for example Soykan [29]. If $m = 2, m = 3$ and $m = 4$, we get the generalized Fibonacci sequence, generalized Tribonacci sequence and generalized Tetranacci sequence, respectively. Next, we recall some information on circulant (r-circulant, geometric circulant) matrices and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm.

Let $n \geq 2$ be an integer and r be any real or complex number. An $n \times n$ matrix C_r is called a r-circulant matrix if it of the form

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

and the r-circulant matrix C_r is denoted by $C_r = Circ_r(c_0, c_1, ..., c_{n-1})$. If $r = 1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = Circ(c_0, c_1, ..., c_{n-1})$.

Circulant matrix was first proposed by Davis in [30]. This matrix has many interesting properties, and it is one of the most important research subject in the field of the computational and pure mathematics (see for example references given in Table 3). For instance, Shen and Cen [31] studied on the norms of r-circulant matrices with Fibonacci and Lucas numbers. Then, later Kızılateş and Tuglu [32] defined a new geometric circulant matrix as follows:

$$C_{r^*} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n\times n}$$

and then they obtained the bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci numbers and Lucas numbers. When the parameter satisfies $r = 1$, we get the classical circulant matrix. See also Polatlı [33] for the spectral norms of r-circulant matrices with a type of Catalan triangle numbers.

The Frobenius (or Euclidean) norm and spectral norm of a matrix $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ are defined respectively as follows:

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} \text{ and } \|A\|_2 = \left(\max_{1\leq i\leq n} |\lambda_i|\right)^{1/2}$$

where λ_i 's are the eigenvalues of the matrix A^*A and A^* is the conjugate of transpose of the matrix A . The maximum column length norm $c_1(.)$ and the maximum row length norm $r_1(.)$ of an matrix of order $n \times n$ are defined as follows:

$$c_1(A) = \max_{1\leq j\leq n}\left(\sum_{i=1}^{n} |a_{ij}|^2\right)^{1/2} \text{ and } r_1(A) = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}.$$

Note that the following inequality holds for any matrix $A = M_{n\times n}(\mathbb{C})$:

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F .$$

Calculations of the above norms $\|A\|_F$, $\|A\|_2$, $c_1(A)$ and $r_1(A)$ require the sum of the squares of the numbers a_{ij}. As in our case, the numbers a_{ij} can be chosen as elements of second, third or higher order linear recurrence sequences.

In the following Table 3, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences which require sum formulas of second powers of numbers in m-step Fibonacci sequences ($m = 2, 3, 4$).

Table 3. Papers on the norms

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[34,32,35,36,37,38,39,31,40,41,42,43]
Pell, Pell-Lucas	[44,45]
Jacobsthal, Jacobsthal-Lucas	[46,47,48,49]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[50,51]
Padovan, Perrin	[52,53,54]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[55]

3 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 3.1. *Let x be a complex number. For $n \geq 0$ we have the following formulas:*

If $(sx+1)\left(r^2x - s^2x^2 + 2sx - 1\right) \neq 0$ then

(a)

$$\sum_{k=0}^{n} kx^k W_k^2 = \frac{\Omega_1}{(sx+1)^2(r^2x - s^2x^2 + 2sx - 1)^2}$$

where

$$\begin{aligned}
\Omega_1 = {} & x^{n+2}(ns^4x^4 - r^2s^2x^3 - nr^2s^2x^3 - 2ns^3x^3 - 2s^2x^2 - 2r^2sx^2 + nr^2x + 2nsx + 4sx + r^2x - n - 2)W_{n+2}^2 \\
& + x^{n+1}(nr^2s^4x^5 - s^4x^4 - 2r^2s^3x^4 - nr^4s^2x^4 - r^4s^2x^4 + ns^4x^4 + 2s^3x^3 - 3nr^2s^2x^3 - 2nr^4sx^3 \\
& - 2ns^3x^3 - 2r^4sx^3 - 2r^2s^2x^3 - nr^4x^2 - r^4x^2 - 2s^2x^2 + 2nsx + 2nr^2x + 2r^2x + 2sx - n - 1)W_{n+1}^2 \\
& + 2rsx^{n+3}(-ns^3x^3 + s^2x^2 + ns^2x^2 + r^2sx^2 + nr^2sx^2 + 2sx + nr^2x + nsx + 2r^2x - n - 3)W_{n+2}W_{n+1} \\
& + x(s^4x^4 - 2s^3x^3 + 2s^2x^2 + 2r^2sx^2 - 2sx + 1)W_1^2 + s^2x^2(r^2s^2x^3 + 2r^2sx^2 + 2s^2x^2 - r^2x - 4sx + 2)W_0^2 \\
& - 2rsx^2(s^3x^3 + r^2x + sx - 2)W_1W_0.
\end{aligned}$$

(b)

$$\sum_{k=0}^{n} kx^k W_{k+1}W_k = \frac{\Omega_2}{(sx+1)^2(r^2x - s^2x^2 + 2sx - 1)^2}$$

where

$$\begin{aligned}\Omega_2 \quad = \quad & (rx^{n+2}(-ns^3x^3+s^3x^3+ns^2x^2+nr^2sx^2+sx+nr^2x+nsx+r^2x-n-2)W_{n+2}^2 \\ & +rs^2x^{n+3}(-ns^3x^3+s^2x^2+ns^2x^2+r^2sx^2+nr^2sx^2+2sx+nr^2x+nsx+2r^2x-n-3)W_{n+1}^2 \\ & -x^{n+1}(-ns^5x^5+s^4x^4+ns^4x^4+2r^2s^3x^4+2r^2s^2x^3+2ns^3x^3+2nr^2s^2x^3+nr^4sx^3+nr^4x^2 \\ & -2ns^2x^2+2r^2sx^2+r^4x^2-2s^2x^2-nsx-2nr^2x-2r^2x+n+1)W_{n+1}W_{n+2} \\ & +rx(-2s^3x^3+s^2x^2+r^2sx^2+1)W_1^2-rs^2x^2(s^3x^3+r^2x+sx-2)W_0^2 \\ & +sx(s^4x^4+2r^2s^2x^3-r^4x^2-2s^2x^2+2r^2x+1)W_0W_1).\end{aligned}$$

Proof. Using the recurrence relation

$$W_{n+2}=rW_{n+1}+sW_n$$

i.e.

$$\begin{aligned} sW_n &= W_{n+2}-rW_{n+1}, \\ s^2W_n^2 &= (W_{n+2}-rW_{n+1})^2=W_{n+2}^2+r^2W_{n+1}^2-2rW_{n+2}W_{n+1}\end{aligned}$$

we obtain

$$\begin{aligned} s^2nx^nW_n^2 &= nx^nW_{n+2}^2+nr^2x^nW_{n+1}^2-2r\times nx^nW_{n+2}W_{n+1} \\ s^2(n-1)x^{n-1}W_{n-1}^2 &= (n-1)x^{n-1}W_{n+1}^2+(n-1)r^2x^{n-1}W_n^2-2r\times(n-1)x^{n-1}W_{n+1}W_n \\ s^2(n-2)x^{n-2}W_{n-2}^2 &= (n-2)x^{n-2}W_n^2+(n-2)r^2x^{n-2}W_{n-1}^2-2r\times(n-2)x^{n-2}W_nW_{n-1} \\ &\vdots \\ s^23x^3W_3^2 &= 3x^3W_5^2+3r^2x^3W_4^2-2r\times 3x^3W_5W_4 \\ s^22x^2W_2^2 &= 2x^2W_4^2+2r^2x^2W_3^2-2r\times 2x^2W_4W_3 \\ s^2x^1W_1^2 &= x^1W_3^2+r^2x^1W_2^2-2r\times x^1W_3W_2 \\ s^2\times 0\times x^0W_0^2 &= 0\times x^0W_2^2+0\times r^2x^0W_1^2-2r\times 0\times x^0W_2W_1\end{aligned}$$

If we add the above equations by side by, we get

$$s^2\sum_{k=0}^{n}kx^kW_k^2=\sum_{k=2}^{n+2}(k-2)x^{k-2}W_k^2+r^2\sum_{k=1}^{n+1}(k-1)x^{k-1}W_k^2-2r\sum_{k=1}^{n+1}(k-1)x^{k-1}W_{k+1}W_k. \tag{3.1}$$

Note that

$$\begin{aligned}\sum_{k=2}^{n+2}(k-2)x^{k-2}W_k^2 &= 2x^{-2}W_0^2+x^{-1}W_1^2+(n-1)x^{n-1}W_{n+1}^2+nx^nW_{n+2}^2 \\ &\quad +x^{-2}(\sum_{k=0}^{n}kx^kW_k^2-2\sum_{k=0}^{n}x^kW_k^2), \\ \sum_{k=1}^{n+1}(k-1)x^{k-1}W_k^2 &= x^{-1}W_0^2+nx^nW_{n+1}^2+x^{-1}(\sum_{k=0}^{n}kx^kW_k^2-\sum_{k=0}^{n}x^kW_k^2), \\ \sum_{k=1}^{n+1}(k-1)x^{k-1}W_{k+1}W_k &= x^{-1}W_1W_0+nx^nW_{n+2}W_{n+1}+x^{-1}(\sum_{k=0}^{n}kx^kW_{k+1}W_k-\sum_{k=0}^{n}x^kW_{k+1}W_k).\end{aligned}$$

If we put them into the (3.1) we get

$$
\begin{aligned}
s^2\sum_{k=0}^{n}kx^kW_k^2 &= nx^nW_{n+2}^2+((n-1)x^{n-1}+r^2nx^n)W_{n+1}^2-2rnx^nW_{n+2}W_{n+1}+x^{-1}W_1^2 \\
&+(2x^{-2}+r^2x^{-1})W_0^2-2rx^{-1}W_1W_0+(x^{-2}+r^2x^{-1})\sum_{k=0}^{n}kx^kW_k^2 \\
&+(-2x^{-2}-r^2x^{-1})\sum_{k=0}^{n}x^kW_k^2-2rx^{-1}\sum_{k=0}^{n}kx^kW_{k+1}W_k+2rx^{-1}\sum_{k=0}^{n}x^kW_{k+1}W_k
\end{aligned} \tag{3.2}
$$

Next we calculate $\sum_{k=1}^{n}kx^kW_{k+1}W_k$. Using the recurrence relation

$$W_{n+2}=rW_{n+1}+sW_n\Rightarrow sW_n=W_{n+2}-rW_{n+1}$$

i.e.

$$sW_n=W_{n+2}-rW_{n+1}$$

and multiplying the both side of the last relations by W_{n+1} we obtain

$$sW_{n+1}W_n=W_{n+2}W_{n+1}-rW_{n+1}^2$$

and so

$$
\begin{aligned}
snx^nW_{n+1}W_n &= nx^nW_{n+2}W_{n+1}-r\times nx^nW_{n+1}^2 \\
s(n-1)x^{n-1}W_nW_{n-1} &= (n-1)x^{n-1}W_{n+1}W_n-r\times(n-1)x^{n-1}W_n^2 \\
s(n-2)Wx_{n-1}^{n-2}W_{n-2} &= (n-2)x^{n-2}W_nW_{n-1}-r(n-2)x^{n-2}W_{n-1}^2 \\
&\vdots \\
s\times 3x^3W_4W_3 &= 3x^3W_5W_4-r\times 3x^3W_4^2 \\
s\times 2x^2W_3W_2 &= 2x^2W_4W_3-r\times 2x^2W_3^2 \\
sx^1W_2W_1 &= x^1W_3W_2-rx^1W_2^2 \\
s\times 0\times x^0W_1W_0 &= 0\times x^0W_2W_1-r\times 0\times x^0W_1^2
\end{aligned}
$$

If we add the above equations by side by, we get

$$s\sum_{k=0}^{n}kx^kW_{k+1}W_k=\sum_{k=1}^{n+1}(k-1)x^{k-1}W_{k+1}W_k-r\sum_{k=1}^{n+1}(k-1)x^{k-1}W_k^2 \tag{3.3}$$

Note that

$$
\begin{aligned}
\sum_{k=1}^{n+1}(k-1)x^{k-1}W_{k+1}W_k &= x^{-1}W_1W_0+nx^nW_{n+2}W_{n+1}+x^{-1}\sum_{k=0}^{n}kx^kW_{k+1}W_k-x^{-1}\sum_{k=0}^{n}x^kW_{k+1}W_k \\
\sum_{k=1}^{n+1}(k-1)x^{k-1}W_k^2 &= x^{-1}W_0^2+nx^nW_{n+1}^2+x^{-1}\sum_{k=0}^{n}kx^kW_k^2-x^{-1}\sum_{k=0}^{n}x^kW_k^2.
\end{aligned}
$$

If we put them in (3.3) we obtain

$$
\begin{aligned}
s\sum_{k=0}^{n}kx^kW_{k+1}W_k &= -rnx^nW_{n+1}^2+nx^nW_{n+2}W_{n+1}-rx^{-1}W_0^2+x^{-1}W_1W_0+x^{-1}\sum_{k=0}^{n}kx^kW_{k+1}W_k \\
&-x^{-1}\sum_{k=0}^{n}x^kW_{k+1}W_k-rx^{-1}\sum_{k=0}^{n}kx^kW_k^2+rx^{-1}\sum_{k=0}^{n}x^kW_k^2
\end{aligned} \tag{3.4}
$$

Then, using

$$W_2=(rW_1+sW_0)$$

and Theorem 1.1 and solving the system (3.2)-(3.4), the required results of (a) and (b) follow.

3.1 The Case $x = 1$

The case $x = 1$ of Theorem 3.1 is given in [18]. In this subsection, we only consider the case $x = 1, r = 1, s = 2$ and we present a theorem which its proof is different than given in [18]).

Observe that setting $x = 1, r = 1, s = 2$ (i.e. for the generalized Jacobsthal case) in Theorem 3.1 (a) and (b) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x = 1, r = 1, s = 2$ then we have the following theorem.

Theorem 3.2. *If $x = 1, r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k^2 = \frac{1}{54}((3n^2 + 9n - 58)W_{n+2}^2 + (12n^2 + 6n - 238)W_{n+1}^2 - 4(3n^2 + 3n - 58)W_{n+1} W_{n+2} + 64W_1^2 + 232W_0^2 - 232W_1W_0).$

(b) $\sum_{k=0}^{n} kW_{k+1}W_k = \frac{1}{54}((-3n^2 + 3n + 58)W_{n+2}^2 - 4(3n^2 + 3n - 58)W_{n+1}^2 + (12n^2 + 18n - 238)W_{n+2}W_{n+1} - 52W_1^2 - 232W_0^2 + 244W_1W_0).$

Proof.

(a) We use Theorem 3.1 (a). If we set $r = 1, s = 2$ in Theorem 3.1 (a) then we have

$$\sum_{k=0}^{n} kx^k W_k^2 = \frac{f_1(x)}{(2x+1)^2 (4x^2 - 5x + 1)^2}$$

where

$$\begin{aligned} f_1(x) = & -x^{n+2}(n - 9x + 20nx^3 - 16nx^4 - 5nx + 12x^2 + 4x^3 + 2)W_{n+2}^2 \\ & -x^{n+1}(n - 6x + nx^2 + 32nx^3 - 12nx^4 - 16nx^5 - 6nx + 9x^2 - 4x^3 + 36x^4 + 1)W_{n+1}^2 \\ & +4x^{n+3}(6x - n + 6nx^2 - 8nx^3 + 3nx + 6x^2 - 3)W_{n+2}W_{n+1} \\ & +x(16x^4 - 16x^3 + 12x^2 - 4x + 1)W_1^2 + 4x^2(4x^3 + 12x^2 - 9x + 2)W_0^2 - 4x^2(8x^3 + 3x - 2)W_1W_0. \end{aligned}$$

For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=0}^{n} kW_k^2 &= \left. \frac{\frac{d}{dx}(f_1(x))}{\frac{d}{dx}((2x+1)^2 (4x^2 - 5x + 1)^2)} \right|_{x=1} \\ &= \frac{1}{54}((3n^2 + 9n - 58)W_{n+2}^2 + (12n^2 + 6n - 238)W_{n+1}^2 \\ &\quad -4(3n^2 + 3n - 58)W_{n+1}W_{n+2} + 64W_1^2 + 232W_0^2 - 232W_1W_0). \end{aligned}$$

(b) We use Theorem 3.1 (b). If we set $r = 1, s = 2$ in Theorem 3.1 (b) then we have

$$\sum_{k=0}^{n} kx^k W_{k+1}W_k = \frac{f_2(x)}{(2x+1)^2 (4x^2 - 5x + 1)^2}$$

where

$$\begin{aligned} f_2(x) = & \; x^{n+2}(3x - n + 6nx^2 - 8nx^3 + 3nx + 8x^3 - 2)W_{n+2}^2 \\ & +4x^{n+3}(6x - n + 6nx^2 - 8nx^3 + 3nx + 6x^2 - 3)W_{n+1}^2 \\ & -x^{n+1}(n - 2x - 7nx^2 + 26nx^3 + 16nx^4 - 32nx^5 - 4nx - 3x^2 + 8x^3 + 32x^4 + 1)W_{n+2}W_{n+1} \\ & +x(-16x^3 + 6x^2 + 1)W_1^2 - 4x^2(8x^3 + 3x - 2)W_0^2 + 2x(16x^4 + 8x^3 - 9x^2 + 2x + 1)W_1W_0 \end{aligned}$$

For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we obtain

$$\begin{aligned}\sum_{k=0}^{n} kW_{k+1}W_k &= \left.\frac{\frac{d}{dx}(f_2(x))}{\frac{d}{dx}((2x+1)^2(4x^2-5x+1)^2)}\right|_{x=1} \\ &= \frac{1}{54}((-3n^2+3n+58)W_{n+2}^2 - 4(3n^2+3n-58)W_{n+1}^2 \\ &\quad +(12n^2+18n-238)W_{n+2}W_{n+1} - 52W_1^2 - 232W_0^2 + 244W_1W_0).\end{aligned}$$

Note that different forms of the sum formulas of the above Theorem (b) and (c) are given in [18].

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 3.3. *For $n \geq 1$, Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} kJ_k^2 = \frac{1}{54}((3n^2+9n-58)J_{n+2}^2 + (12n^2+6n-238)J_{n+1}^2 - 4(3n^2+3n-58)J_{n+1}J_{n+2} + 64)$.

(b) $\sum_{k=0}^{n} kJ_{k+1}J_k = \frac{1}{54}((-3n^2+3n+58)J_{n+2}^2 - 4(3n^2+3n-58)J_{n+1}^2 + (12n^2+18n-238)J_{n+2}J_{n+1} - 52)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.4. *For $n \geq 1$, Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kj_k^2 = \frac{1}{54}((3n^2+9n-58)j_{n+2}^2 + (12n^2+6n-238)j_{n+1}^2 - 4(3n^2+3n-58)j_{n+1}j_{n+2} + 528)$.

(b) $\sum_{k=0}^{n} kj_{k+1}j_k = \frac{1}{54}((-3n^2+3n+58)j_{n+2}^2 - 4(3n^2+3n-58)j_{n+1}^2 + (12n^2+18n-238)j_{n+2}j_{n+1} - 492)$.

3.2 The Case $x = -1$

We now consider the case $x = -1$ in Theorem 3.1. The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Observe that setting $x = -1, r = 1, s = 1$ in Theorem 3.1 (a) and (b) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x = -1, r = 1, s = 1$ then we have the following theorem.

Taking $x = -1, r = 1, s = 1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Theorem 3.5. *If $x = -1, r = 1, s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{25}((-1)^n((5n^2+10n-31)W_{n+2}^2 + (-5n^2+5n+41)W_{n+1}^2 - (5n^2+15n-26)W_{n+2}W_{n+1}) - 36W_1^2 + 31W_0^2 + 36W_1W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{50}((-1)^n((5n^2+5n-36)W_{n+2}^2 - (5n^2+15n-26)W_{n+1}^2 + (-5n^2+15n+46)W_{n+2}W_{n+1}) - 36W_1^2 + 36W_0^2 + 26W_1W_0)$.

Proof.

(a) We use Theorem 3.1 (a). If we set $r = 1, s = 1$ in Theorem 3.1 (a) then we have

$$\sum_{k=0}^{n} kx^k W_k^2 = \frac{g_1(x)}{(x+1)^2(x^2-3x+1)^2}$$

where

$$\begin{aligned} g_1(x) &= -x^{n+2}(n-5x+3nx^3-nx^4-3nx+4x^2+x^3+2)W_{n+2}^2 \\ &\quad -x^{n+1}(n-4x+nx^2+7nx^3-nx^5-4nx+3x^2+2x^3+4x^4+1)W_{n+1}^2 \\ &\quad +2x^{n+3}(4x-n+2nx^2-nx^3+2nx+2x^2-3)W_{n+2}W_{n+1} \\ &\quad +x(x^4-2x^3+4x^2-2x+1)W_1^2+x^2(x^3+4x^2-5x+2)W_0^2-2x^2(x^3+2x-2)W_1W_0. \end{aligned}$$

For $x=-1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=0}^{n} k(-1)^k W_k^2 &= \left.\frac{\frac{d}{dx}(g_1(x))}{\frac{d}{dx}((x+1)^2(x^2-3x+1)^2)}\right|_{x=-1} \\ &= \frac{1}{25}((-1)^n((5n^2+10n-31)W_{n+2}^2+(-5n^2+5n+41)W_{n+1}^2 \\ &\quad -(5n^2+15n-26)W_{n+2}W_{n+1})-36W_1^2+31W_0^2+36W_1W_0). \end{aligned}$$

(b) We use Theorem 3.1 (b). If we set $r=1, s=1$ in Theorem 3.1 (b) then we have

$$\sum_{k=0}^{n} kx^k W_{k+1}W_k = \frac{g_2(x)}{(x+1)^2(x^2-3x+1)^2}$$

where

$$\begin{aligned} g_2(x) &= x^{n+2}(2x-n+2nx^2-nx^3+2nx+x^3-2)W_{n+2}^2 \\ &\quad +x^{n+3}(4x-n+2nx^2-nx^3+2nx+2x^2-3)W_{n+1}^2 \\ &\quad -x^{n+1}(n-2x-nx^2+5nx^3+nx^4-nx^5-3nx+x^2+2x^3+3x^4+1)W_{n+2}W_{n+1} \\ &\quad +x(-2x^3+2x^2+1)W_1^2-x^2(x^3+2x-2)W_0^2+x(x^4+2x^3-3x^2+2x+1)W_1W_0. \end{aligned}$$

For $x=-1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we obtain

$$\begin{aligned} \sum_{k=0}^{n} k(-1)^k W_{k+1}W_k &= \left.\frac{\frac{d}{dx}(g_2(x))}{\frac{d}{dx}((x+1)^2(x^2-3x+1)^2)}\right|_{x=-1} \\ &= \frac{1}{50}((-1)^n((5n^2+5n-36)W_{n+2}^2-(5n^2+15n-26)W_{n+1}^2 \\ &\quad +(-5n^2+15n+46)W_{n+2}W_{n+1})-36W_1^2+36W_0^2+26W_1W_0). \end{aligned}$$

From the above theorem, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n=F_n$ with $F_0=0, F_1=1$).

Corollary 3.6. *For $n\geq 0$, Fibonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k F_k^2 = \frac{1}{25}((-1)^n((5n^2+10n-31)F_{n+2}^2+(-5n^2+5n+41)F_{n+1}^2-(5n^2+15n-26)F_{n+2}F_{n+1})-36)$.

(b) $\sum_{k=0}^{n} k(-1)^k F_{k+1}F_k = \frac{1}{50}((-1)^n((5n^2+5n-36)F_{n+2}^2-(5n^2+15n-26)F_{n+1}^2+(-5n^2+15n+46)F_{n+2}F_{n+1})-36)$.

Taking $W_n=L_n$ with $L_0=2, L_1=1$ in the last theorem, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.7. *For $n\geq 0$, Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k L_k^2 = \frac{1}{25}((-1)^n((5n^2+10n-31)L_{n+2}^2 + (-5n^2+5n+41)L_{n+1}^2 - (5n^2+15n-26)L_{n+2}L_{n+1}) + 160)$.

(b) $\sum_{k=0}^{n} k(-1)^k L_{k+1}L_k = \frac{1}{50}((-1)^n((5n^2+5n-36)L_{n+2}^2 - (5n^2+15n-26)L_{n+1}^2 + (-5n^2+15n+46)L_{n+2}L_{n+1}) + 160)$.

Note that setting $x=-1, r=2, s=1$ in Theorem 3.1 (a) and (b) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x=-1, r=2, s=1$ then we have the following theorem.

Taking $x=-1, r=2, s=1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Theorem 3.8. *If $x=-1, r=2, s=1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{32}((-1)^n((4n^2+8n-23)W_{n+2}^2 + (-4n^2+16n+43)W_{n+1}^2 - (8n^2+24n-38)W_{n+1}W_{n+2}) - 27W_1^2 + 23W_0^2 + 54W_1W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{32}((-1)^n((4n^2+4n-27)W_{n+2}^2 - (4n^2+12n-19)W_{n+1}^2 - (8n^2-58)W_{n+2}W_{n+1}) - 27W_1^2 + 27W_0^2 + 50W_1W_0)$.

Proof.

(a) We use Theorem 3.1 (a). If we set $r=2, s=1$ in Theorem 3.1 (a) then we have

$$\sum_{k=0}^{n} kx^k W_k^2 = \frac{h_1(x)}{(x+1)^2(x^2-6x+1)^2}$$

where

$$\begin{aligned} h_1(x) = & -x^{n+2}(n-8x+6nx^3-nx^4-6nx+10x^2+4x^3+2)W_{n+2}^2 \\ & -x^{n+1}(n-10x+16nx^2+46nx^3+15nx^4-4nx^5-10nx+18x^2+38x^3+25x^4+1)W_{n+1}^2 \\ & +4x^{n+3}(10x-n+5nx^2-nx^3+5nx+5x^2-3)W_{n+2}W_{n+1} + x(x^4-2x^3+10x^2-2x+1)W_1^2 \\ & +x^2(4x^3+10x^2-8x+2)W_0^2 - 4x^2(x^3+5x-2)W_1W_0. \end{aligned}$$

For $x=-1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=0}^{n} k(-1)^k W_k^2 &= \left.\frac{\frac{d}{dx}(h_1(x))}{\frac{d}{dx}((x+1)^2(x^2-6x+1)^2)}\right|_{x=-1} \\ &= \frac{1}{32}((-1)^n((4n^2+8n-23)W_{n+2}^2 + (-4n^2+16n+43)W_{n+1}^2 \\ & \quad -(8n^2+24n-38)W_{n+1}W_{n+2}) - 27W_1^2 + 23W_0^2 + 54W_1W_0). \end{aligned}$$

(b) We use Theorem 3.1 (b). If we set $r=2, s=1$ in Theorem 3.1 (b) then we have

$$\sum_{k=0}^{n} kx^k W_{k+1}W_k = \frac{h_2(x)}{(x+1)^2(x^2-6x+1)^2}$$

where

$$\begin{aligned} h_2(x) = & \; 2x^{n+2}(5x-n+5nx^2-nx^3+5nx+x^3-2)W_{n+2}^2 \\ & +2x^{n+3}(10x-n+5nx^2-nx^3+5nx+5x^2-3)W_{n+1}^2 \\ & -x^{n+1}(n-8x+14nx^2+26nx^3+nx^4-nx^5-9nx+22x^2+8x^3+9x^4+1)W_{n+2}W_{n+1} \\ & +2x(-2x^3+5x^2+1)W_1^2 - 2x^2(x^3+5x-2)W_0^2 + x(x^4+8x^3-18x^2+8x+1)W_1W_0 \end{aligned}$$

For $x=-1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we obtain

$$\begin{aligned}\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k &= \left.\frac{\frac{d}{dx}(h_2(x))}{\frac{d}{dx}((x+1)^2(x^2-6x+1)^2)}\right|_{x=-1} \\ &= \frac{1}{32}((-1)^n((4n^2+4n-27)W_{n+2}^2-(4n^2+12n-19)W_{n+1}^2 \\ &\quad -(8n^2-58)W_{n+2}W_{n+1})-27W_1^2+27W_0^2+50W_1W_0).\end{aligned}$$

From the last theorem, we have the following corollary which gives sum formulas of Pell numbers (take $W_n=P_n$ with $P_0=0, P_1=1$).

Corollary 3.9. *For $n\geq 1$, Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k P_k^2 = \frac{1}{32}((-1)^n((4n^2+8n-23)P_{n+2}^2+(-4n^2+16n+43)P_{n+1}^2-(8n^2+24n-38)P_{n+1}P_{n+2})-27)$.

(b) $\sum_{k=0}^{n} k(-1)^k P_{k+1}P_k = \frac{1}{32}((-1)^n((4n^2+4n-27)P_{n+2}^2-(4n^2+12n-19)P_{n+1}^2-(8n^2-58)P_{n+2}P_{n+1})-27)$.

Taking $W_n=Q_n$ with $Q_0=2, Q_1=2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.10. *For $n\geq 1$, Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k Q_k^2 = \frac{1}{32}((-1)^n((4n^2+8n-23)Q_{n+2}^2+(-4n^2+16n+43)Q_{n+1}^2-(8n^2+24n-38)Q_{n+1}Q_{n+2})+200)$.

(b) $\sum_{k=0}^{n} k(-1)^k Q_{k+1}Q_k = \frac{1}{32}((-1)^n((4n^2+4n-27)Q_{n+2}^2-(4n^2+12n-19)Q_{n+1}^2-(8n^2-58)Q_{n+2}Q_{n+1})+200)$.

Taking $x=-1, r=1, s=2$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Proposition 3.1. *If $x=-1, r=1, s=2$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(-1)^k W_k^2 = \frac{1}{100}((-1)^n((30n-19)W_{n+2}^2-(20n-56)W_{n+1}^2-4(10n-3)W_{n+2}W_{n+1})-49W_1^2+76W_0^2+52W_1W_0)$.

(b) $\sum_{k=0}^{n} k(-1)^k W_{k+1}W_k = \frac{1}{100}((-1)^n((10n-13)W_{n+2}^2-4(10n-3)W_{n+1}^2+(20n+24)W_{n+2}W_{n+1})-23W_1^2+52W_0^2+4W_1W_0)$.

From the last proposition we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1$).

Corollary 3.11. *For $n\geq 1$, Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k J_k^2 = \frac{1}{100}((-1)^n((30n-19)J_{n+2}^2-(20n-56)J_{n+1}^2-4(10n-3)J_{n+2}J_{n+1})-49)$.

(b) $\sum_{k=0}^{n} k(-1)^k J_{k+1}J_k = \frac{1}{100}((-1)^n((10n-13)J_{n+2}^2-4(10n-3)J_{n+1}^2+(20n+24)J_{n+2}J_{n+1})-23)$.

Taking $W_n=j_n$ with $j_0=2, j_1=1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.12. *For $n\geq 1$, Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} k(-1)^k j_k^2 = \frac{1}{100}((-1)^n((30n-19)j_{n+2}^2-(20n-56)j_{n+1}^2-4(10n-3)j_{n+2}j_{n+1})+359)$.

(b) $\sum_{k=0}^{n} k(-1)^k j_{k+1}j_k = \frac{1}{100}((-1)^n((10n-13)j_{n+2}^2-4(10n-3)j_{n+1}^2+(20n+24)j_{n+2}j_{n+1})+193)$.

3.3 The Case $x = 1 + i$

We now consider the complex case $x = 1 + i$ in Theorem 3.1. The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Taking $x = 1 + i, r = 1, s = 1$ in Theorem 3.1 we obtain the following proposition.

Proposition 3.2. *If $x = 1 + i, r = s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} k(1+i)^k W_k^2 = \frac{1}{-7+24i}((1+i)^n (2((3+4i)n+5+5i)W_{n+2}^2+((29-3i)n+29+17i)W_{n+1}^2 - 4((7+i)n+9+7i)W_{n+2}W_{n+1}) - (3-i)W_1^2 - (10+10i)W_0^2 + (16+8i)W_1W_0)$.

(b) $\sum_{k=0}^{n} k(1+i)^k W_{k+1}W_k = \frac{1}{-7+24i}((1+i)^n (-2((4-3i)n+4+2i)W_{n+2}^2 - 2((7+i)n+9+7i) W_{n+1}^2 + ((21+3i)n+21+13i)W_{n+2}W_{n+1}) + (5+5i)W_1^2 + (8+4i)W_0^2 - (5+5i)W_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.13. *For $n \geq 0$, Fibonacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} k(1+i)^k F_k^2 = \frac{1}{-7+24i}((1+i)^n (2((3+4i)n+5+5i)F_{n+2}^2+((29-3i)n+29+17i)F_{n+1}^2 - 4((7+i)n+9+7i)F_{n+2}F_{n+1}) - 3+i)$.

(b) $\sum_{k=0}^{n} k(1+i)^k F_{k+1}F_k = \frac{1}{-7+24i}((1+i)^n (-2((4-3i)n+4+2i)F_{n+2}^2 - 2((7+i)n+9+7i) F_{n+1}^2 + ((21+3i)n+21+13i)F_{n+2}F_{n+1}) + 5+5i)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.14. *For $n \geq 0$, Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} k(1+i)^k L_k^2 = \frac{1}{-7+24i}((1+i)^n (2((3+4i)n+5+5i)L_{n+2}^2+((29-3i)n+29+17i)L_{n+1}^2 - 4((7+i)n+9+7i)L_{n+2}L_{n+1}) - 11-23i)$.

(b) $\sum_{k=0}^{n} k(1+i)^k L_{k+1}L_k = \frac{1}{-7+24i}((1+i)^n (-2((4-3i)n+4+2i)L_{n+2}^2 - 2((7+i)n+9+7i) L_{n+1}^2 + ((21+3i)n+21+13i)L_{n+2}L_{n+1}) + 27+11i)$.

Corresponding sums of the other second order linear sequences can be calculated similarly when $x = 1 + i$.

4 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 4.1. *Let x be a complex number. For $n \geq 1$ we have the following formulas:*
If $(s+x)\left(r^2x + 2sx - s^2 - x^2\right) \neq 0$ then

(a)

$$\sum_{k=1}^{n} kx^k W_{-k}^2 = \frac{\Omega_3}{(s+x)^2 \left(r^2x + 2sx - s^2 - x^2\right)^2}$$

where

$$\begin{aligned}
\Omega_3 &= x^{n+1}(-x^4+nx^4+2sx^3-2nsx^3-nr^2x^3-2r^2sx^2 \\
&-2s^2x^2+2ns^3x+nr^2s^2x+2s^3x-s^4-ns^4)W_{-n+1}^2 \\
&+x^{n+1}(nx^5-2nsx^4-2nr^2x^4-2s^2x^3+nr^4x^3+4s^3x^2+r^2s^2x^2+2ns^3x^2 \\
&+3nr^2s^2x^2+2nr^4sx^2+nr^4s^2x-2r^2s^3x-ns^4x-2s^4x-r^2s^4-nr^2s^4)W_{-n}^2 \\
&-2rsx^{n+1}(2x^3-nx^3-sx^2-r^2x^2+nr^2x^2+nsx^2+ns^2x+nr^2sx-ns^3-s^3)W_{-n+1}W_{-n} \\
&+x(x^4-2sx^3-2s^3x+2s^2x^2+2r^2sx^2+s^4)W_1^2 \\
&+s^2x(-4sx^2+2s^2x+r^2s^2-r^2x^2+2x^3+2r^2sx)W_0^2-2rsx(sx^2+r^2x^2+s^3-2x^3)W_1W_0.
\end{aligned}$$

(b)

$$\sum_{k=1}^{n} kx^k W_{-k+1}W_{-k} = \frac{\Omega_4}{(s+x)^2\left(r^2x+2sx-s^2-x^2\right)^2}$$

where

$$\begin{aligned}
\Omega_4 &= rx^{n+2}(nx^3-x^3-nsx^2-nr^2x^2-nr^2sx-ns^2x-r^2sx-s^2x+2s^3+ns^3)W_{-n+1}^2 \\
&+rs^2x^{n+1}(nx^3-2x^3-nsx^2+sx^2+r^2x^2-nr^2x^2-ns^2x-nr^2sx+s^3+ns^3)W_{-n}^2 \\
&+x^{n+1}(nx^5-sx^4-nsx^4-2nr^2x^4+nr^4x^3-2ns^2x^3-2r^2sx^3+r^4sx^2+2ns^3x^2+2s^3x^2 \\
&+2nr^2s^2x^2+nr^4sx^2-2r^2s^3x+ns^4x-s^5-ns^5)W_{-n+1}W_{-n}+rx^2(x^3+s^2x+r^2sx-2s^3)W_1^2 \\
&-rs^2x(r^2x^2+s^3+sx^2-2x^3)W_0^2+sx(2r^2x^3-r^4x^2-2s^2x^2+s^4+x^4+2r^2s^2x)W_1W_0.
\end{aligned}$$

Proof. Using the recurrence relation

$$W_{-n+2} = r\times W_{-n+1} + s\times W_{-n}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

and using

$$s^2W_{-n}^2 = (W_{-n+2}-rW_{-n+1})^2 = W_{-n+2}^2 + r^2W_{-n+1}^2 - 2rW_{-n+2}W_{-n+1}$$

we obtain

$$\begin{aligned}
s^2nx^nW_{-n}^2 &= nx^nW_{-n+2}^2 + r^2nx^nW_{-n+1}^2 - 2r\times nx^nW_{-n+2}W_{-n+1} \\
s^2(n-1)x^{n-1}W_{-n+1}^2 &= (n-1)x^{n-1}W_{-n+3}^2 + r^2(n-1)x^{n-1}W_{-n+2}^2 - 2r(n-1)x^{n-1}W_{-n+3}W_{-n+2} \\
s^2(n-2)x^{n-2}W_{-n+2}^2 &= (n-2)x^{n-2}W_{-n+4}^2 + r^2(n-2)x^{n-2}W_{-n+3}^2 - 2r(n-2)x^{n-2}W_{-n+4}W_{-n+3} \\
&\vdots \\
s^2\times 4x^4W_{-4}^2 &= 4x^4W_{-2}^2 + x^4r^24W_{-3}^2 - 2r\times 4x^4W_{-2}W_{-3} \\
s^2\times 3x^3W_{-3}^2 &= 3x^3W_{-1}^2 + r^23x^3W_{-2}^2 - 2r\times 3x^3W_{-1}W_{-2} \\
s^2\times 2x^2W_{-2}^2 &= 2x^2W_0^2 + x^2r^22W_{-1}^2 - 2r\times 2x^2W_0W_{-1} \\
s^2x^1W_{-1}^2 &= x^1W_1^2 + r^2x^1W_0^2 - 2rx^1W_1W_0
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
s^2\sum_{k=1}^{n} kx^kW_{-k}^2 &= x^1W_1^2 + x\left(2x+r^2\right)W_0^2 - 2rx^1W_1W_0 - (n+1)x^{n+1}W_{-n+1}^2 \qquad (4.1) \\
&-x^{n+1}\left(2x+nr^2+nx+r^2\right)W_{-n}^2 + 2r(n+1)x^{n+1}W_{-n+1}W_{-n} \\
&+(x^2+r^2x)\sum_{k=1}^{n} kx^kW_{-k}^2 + (2x^2+r^2x)\sum_{k=1}^{n} x^kW_{-k}^2 - 2rx\sum_{k=1}^{n} kx^kW_{-k+1}W_{-k} \\
&-2rx\sum_{k=1}^{n} x^kW_{-k+1}W_{-k}
\end{aligned}$$

Next we calculate $\sum_{k=1}^{n} kW_{-k+1}W_{-k}$. Using the recurrence relation

$$W_{-n+2} = r \times W_{-n+1} + s \times W_{-n}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

and and multiplying the both side of the last relations by W_{-n+1} we get

$$sW_{-n+1}W_{-n} = W_{-n+2}W_{-n+1} - rW_{-n+1}^2$$

and so we obtain

$$\begin{aligned}
s \times nx^n W_{-n+1}W_{-n} &= nx^n W_{-n+2}W_{-n+1} - r \times nx^n W_{-n+1}^2 \\
s \times (n-1)x^{n-1}W_{-n+2}W_{-n+1} &= (n-1)x^{n-1}W_{-n+3}W_{-n+2} - r \times (n-1)x^{n-1}W_{-n+2}^2 \\
s \times (n-2)x^{n-2}W_{-n+3}W_{-n+2} &= (n-2)x^{n-2}W_{-n+4}W_{-n+3} - r \times (n-2)x^{n-2}W_{-n+3}^2 \\
&\vdots \\
s \times 4x^4 W_{-3}W_{-4} &= 4x^4 W_{-2}W_{-3} - r \times 4x^4 W_{-3}^2 \\
s \times 3x^3 W_{-2}W_{-3} &= 3x^3 W_{-1}W_{-2} - r \times 3x^3 W_{-2}^2 \\
s \times 2x^2 W_{-1}W_{-2} &= 2x^2 W_0 W_{-1} - r \times 2x^2 W_{-1}^2 \\
sxW_0W_{-1} &= xW_1W_0 - r \times xW_0^2
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
s\sum_{k=1}^{n} kx^k W_{-k+1}W_{-k} &= -rxW_0^2 + xW_1W_0 + r(n+1)x^{n+1}W_{-n}^2 - (n+1)x^{n+1}W_{-n+1}W_{-n} \qquad (4.2) \\
&\quad + x\sum_{k=1}^{n} kx^k W_{-k+1}W_{-k} + x\sum_{k=1}^{n} x^k W_{-k+1}W_{-k} - rx\sum_{k=1}^{n} kx^k W_{-k}^2 - rx\sum_{k=1}^{n} x^k W_{-k}^2 \qquad (4.3)
\end{aligned}$$

Then, using Theorem 1.2 and solving the system (4.1)-(4.2), the required results of (a) and (b) follow.

4.1 The Case $x = 1$

The case $x = 1$ of Theorem 4.1 is given in [18]. In this subsection, we only consider the case $x = 1, r = 1, s = 2$ and we present a theorem which its proof is different than given in [18] (in fact the formulas given in the following theorem are in different forms than given in [18]).

Observe that setting $x = 1, r = 1, s = 2$ (i.e. for the generalized Jacobsthal case) in Theorem 4.1 (a) and (b) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x = 1, r = 1, s = 2$ then we have the following theorem.

Theorem 4.2. *If $x = 1, r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k}^2 = \frac{1}{54}((3n^2 - 3n - 4)W_{-n+1}^2 + (12n^2 + 18n + 8)W_{-n}^2 - 4(3n^2 + 3n + 2)W_{-n+1}W_{-n} + 4W_1^2 - 8W_0^2 + 8W_1W_0)$.

(b) $\sum_{k=1}^{n} kW_{-k+1}W_{-k} = \frac{1}{54}(-(3n^2 + 9n + 8)W_{-n+1}^2 - 4(3n^2 + 3n + 2)W_{-n}^2 + (12n^2 + 6n - 4)W_{-n+1}W_{-n} + 8W_1^2 + 8W_0^2 + 4W_1W_0)$.

Proof.

(a) We use Theorem 4.1 (a). If we set $r = 1, s = 2$ in Theorem 4.1 (a) then we have

$$\sum_{k=1}^{n} kx^k W_{-k}^2 = \frac{f_3(x)}{(x+2)^2 (x^2-5x+4)^2}$$

where

$$\begin{aligned} f_3(x) &= -x^{n+1}(16n - 16x + 5nx^3 - nx^4 - 20nx + 12x^2 - 4x^3 + x^4 + 16)W_{-n+1}^2 \\ &\quad -x^{n+1}(16n + 48x - 32nx^2 - nx^3 + 6nx^4 - nx^5 + 12nx - 36x^2 + 8x^3 + 16)W_{-n}^2 \\ &\quad +4x^{n+1}(8n - 3nx^2 + nx^3 - 6nx + 3x^2 - 2x^3 + 8)W_{-n+1}W_{-n} \\ &\quad +x(x^4 - 4x^3 + 12x^2 - 16x + 16)W_1^2 \\ &\quad +4x\left(2x^3 - 9x^2 + 12x + 4\right)W_0^2 - 4x\left(-2x^3 + 3x^2 + 8\right)W_1W_0. \end{aligned}$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned} \sum_{k=0}^{n} kW_{-k}^2 &= \left.\frac{\frac{d}{dx}(f_3(x))}{\frac{d}{dx}((x+2)^2(x^2-5x+4)^2)}\right|_{x=1} \\ &= \frac{1}{54}((3n^2 - 3n - 4)W_{-n+1}^2 + (12n^2 + 18n + 8)W_{-n}^2 \\ &\quad -4(3n^2 + 3n + 2)W_{-n+1}W_{-n} + 4W_1^2 - 8W_0^2 + 8W_1W_0). \end{aligned}$$

(b) We use Theorem 4.1 (b). If we set $r = 1, s = 2$ in Theorem 4.1 (b) then we have

$$\sum_{k=1}^{n} kx^k W_{-k+1}W_{-k} = \frac{f_4(x)}{(x+2)^2 (x^2-5x+4)^2}$$

where

$$\begin{aligned} f_4(x) &= -x^{n+2}(6x - 8n + 3nx^2 - nx^3 + 6nx + x^3 - 16)W_{-n+1}^2 \\ &\quad +4x^{n+1}(8n - 3nx^2 + nx^3 - 6nx + 3x^2 - 2x^3 + 8)W_{-n}^2 \\ &\quad -x^{n+1}(32n + 16x - 26nx^2 + 7nx^3 + 4nx^4 - nx^5 - 16nx - 18x^2 + 4x^3 + 2x^4 + 32)W_{-n+1}W_{-n} \\ &\quad +x^2\left(x^3 + 6x - 16\right)W_1^2 - 4x\left(-2x^3 + 3x^2 + 8\right)W_0^2 + 2x\left(x^4 + 2x^3 - 9x^2 + 8x + 16\right)W_1W_0. \end{aligned}$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned} \sum_{k=1}^{n} kW_{-k+1}W_{-k} &= \left.\frac{\frac{d}{dx}(f_4(x))}{\frac{d}{dx}((x+2)^2(x^2-5x+4)^2)}\right|_{x=1} \\ &= \frac{1}{54}(-(3n^2 + 9n + 8)W_{-n+1}^2 - 4(3n^2 + 3n + 2)W_{-n}^2 \\ &\quad +(12n^2 + 6n - 4)W_{-n+1}W_{-n} + 8W_1^2 + 8W_0^2 + 4W_1W_0). \end{aligned}$$

Note that different forms of the sum formulas of the above Theorem (b) and (c) are given in [18].

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 4.3. *For $n \geq 1$, Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n} kJ_{-k}^2 = \frac{1}{54}((3n^2 - 3n - 4)J_{-n+1}^2 + (12n^2 + 18n + 8)J_{-n}^2 - 4(3n^2 + 3n + 2)J_{-n+1}J_{-n} + 4)$.

(b) $\sum_{k=1}^{n} kJ_{-k+1}J_{-k} = \frac{1}{54}(-(3n^2+9n+8)J_{-n+1}^2 - 4(3n^2+3n+2)J_{-n}^2 + (12n^2+6n-4)J_{-n+1}J_{-n} + 8)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 4.4. *For $n \geq 1,$ Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kj_{-k}^2 = \frac{1}{54}((3n^2-3n-4)j_{-n+1}^2 + (12n^2+18n+8)j_{-n}^2 - 4(3n^2+3n+2)j_{-n+1}j_{-n} - 12).$

(b) $\sum_{k=1}^{n} kj_{-k+1}j_{-k} = \frac{1}{54}(-(3n^2+9n+8)j_{-n+1}^2 - 4(3n^2+3n+2)j_{-n}^2 + (12n^2+6n-4)j_{-n+1}j_{-n} + 48).$

4.2 The Case $x = -1$

We now consider the case $x = -1$ in Theorem 4.1.

Observe that setting $x = -1, r = 1, s = 1$ in Theorem 4.1 (a) and (b) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x = -1, r = 1, s = 1$ then we have the following theorem.

Theorem 4.5. *If $x = -1, r = 1, s = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^k W_{-k}^2 = \frac{1}{25}((-1)^n((36-5n^2)W_{-n+1}^2 + (5n^2+15n-31)W_{-n}^2 - (-5n^2+5n+36)W_{-n+1}W_{-n}) - 36W_1^2 + 31W_0^2 + 36W_1W_0).$

(b) $\sum_{k=1}^{n} k(-1)^k W_{-k+1}W_{-k} = \frac{1}{50}((-1)^n(-(5n^2+5n-36)W_{-n+1}^2 + (5n^2-5n-36)W_{-n}^2 + (5n^2+25n-26)W_{-n+1}W_{-n}) - 36W_1^2 + 36W_0^2 + 26W_1W_0).$

Proof.

(a) We use Theorem 4.1 (a). If we set $r = 1, s = 1$ in Theorem 4.1 (a) then we have

$$\sum_{k=1}^{n} kx^k W_{-k}^2 = \frac{g_3(x)}{(x+1)^2(x^2-3x+1)^2}$$

where

$$\begin{aligned} g_3(x) = \; & -x^{n+1}(n-2x+3nx^3-nx^4-3nx+4x^2-2x^3+x^4+1)W_{-n+1}^2 \\ & -x^{n+1}(n+4x-7nx^2-nx^3+4nx^4-nx^5-5x^2+2x^3+1)W_{-n}^2 \\ & +2x^{n+1}(n-2nx^2+nx^3-2nx+2x^2-2x^3+1)W_{-n+1}W_{-n} \\ & +x(x^4-2x^3+4x^2-2x+1)W_1^2 \\ & +x\left(2x^3-5x^2+4x+1\right)W_0^2 - 2x\left(-2x^3+2x^2+1\right)W_1W_0. \end{aligned}$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned} \sum_{k=1}^{n} k(-1)^k W_{-k}^2 &= \left.\frac{\frac{d}{dx}(g_3(x))}{\frac{d}{dx}((x+1)^2(x^2-3x+1)^2)}\right|_{x=-1} \\ &= \frac{1}{25}((-1)^n((36-5n^2)W_{-n+1}^2 + (5n^2+15n-31)W_{-n}^2 \\ &\quad -(-5n^2+5n+36)W_{-n+1}W_{-n}) - 36W_1^2 + 31W_0^2 + 36W_1W_0). \end{aligned}$$

(b) We use Theorem 4.1 (b). If we set $r = 1, s = 1$ in Theorem 4.1 (b) then we have

$$\sum_{k=1}^{n} kx^k W_{-k+1}W_{-k} = \frac{g_4(x)}{(x+1)^2(x^2-3x+1)^2}$$

where

$$\begin{aligned} g_4(x) &= -x^{n+2}(2x-n+2nx^2-nx^3+2nx+x^3-2)W_{-n+1}^2 \\ &+x^{n+1}(n-2nx^2+nx^3-2nx+2x^2-2x^3+1)W_{-n}^2 \\ &-x^{n+1}(n+2x-5nx^2+nx^3+3nx^4-nx^5-nx-3x^2+2x^3+x^4+1)W_{-n+1}W_{-n} \\ &+x^2(x^3+2x-2)W_1^2-x\left(-2x^3+2x^2+1\right)W_0^2+x\left(x^4+2x^3-3x^2+2x+1\right)W_1W_0. \end{aligned}$$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned} \sum_{k=1}^{n} k(-1)^k W_{-k+1}W_{-k} &= \left.\frac{\frac{d}{dx}(g_4(x))}{\frac{d}{dx}((x+1)^2(x^2-3x+1)^2)}\right|_{x=-1} \\ &= \frac{1}{50}((-1)^n\left(-(5n^2+5n-36)W_{-n+1}^2+(5n^2-5n-36)W_{-n}^2\right. \\ &\left.+(5n^2+25n-26)W_{-n+1}W_{-n}\right)-36W_1^2+36W_0^2+26W_1W_0). \end{aligned}$$

From the last theorem, we have the following corollary which gives sum formula of Fibonacci numbers (take $W_n=F_n$ with $F_0=0, F_1=1$).

Corollary 4.6. *For $n\geq 1$, Fibonacci numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^k F_{-k}^2 = \frac{1}{25}((-1)^n\left((36-5n^2)F_{-n+1}^2+(5n^2+15n-31)F_{-n}^2-(-5n^2+5n+36)F_{-n+1}F_{-n}\right)-36)$.

(b) $\sum_{k=1}^{n} k(-1)^k F_{-k+1}F_{-k} = \frac{1}{50}((-1)^n\left(-(5n^2+5n-36)F_{-n+1}^2+(5n^2-5n-36)F_{-n}^2+(5n^2+25n-26)F_{-n+1}F_{-n}\right)-36)$.

Taking $W_n=L_n$ with $L_0=2, L_1=1$ in the last theorem, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 4.7. *For $n\geq 1$, Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^k L_{-k}^2 = \frac{1}{25}((-1)^n\left((36-5n^2)L_{-n+1}^2+(5n^2+15n-31)L_{-n}^2-(-5n^2+5n+36)L_{-n+1}L_{-n}\right)+160)$.

(b) $\sum_{k=1}^{n} k(-1)^k L_{-k+1}L_{-k} = \frac{1}{50}((-1)^n\left(-(5n^2+5n-36)L_{-n+1}^2+(5n^2-5n-36)L_{-n}^2+(5n^2+25n-26)L_{-n+1}L_{-n}\right)+160)$.

Note that setting $x=-1, r=2, s=1$ in Theorem 4.1 (a) and (b) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule (using twice) however provides the evaluation of the sum formulas. If $x=-1, r=2, s=1$ then we have the following theorem.

Theorem 4.8. *If $x=-1, r=2, s=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^k W_{-k}^2 = \frac{1}{32}((-1)^n\left((-4n^2+27)W_{-n+1}^2+(4n^2+24n-23)W_{-n}^2+(8n^2-8n-54)W_{-n+1}W_{-n}\right)-27W_1^2+23W_0^2+54W_1W_0)$.

(b) $\sum_{k=1}^{n} k(-1)^k W_{-k+1}W_{-k} = \frac{1}{32}((-1)^n\left((-4n^2-4n+27)W_{-n+1}^2+(4n^2-4n-27)W_{-n}^2+(8n^2+16n-50)W_{-n+1}W_{-n}\right)-27W_1^2+27W_0^2+50W_1W_0)$.

Proof.

(a) We use Theorem 4.1 (a). If we set $r=2, s=1$ in Theorem 4.1 (a) then we have

$$\sum_{k=1}^{n} kx^k W_{-k}^2 = \frac{h_3(x)}{(x+1)^2(x^2-6x+1)^2}$$

where

$$\begin{aligned} h_3(x) &= -x^{n+1}(n-2x+6nx^3-nx^4-6nx+10x^2-2x^3+x^4+1)W^2_{-n+1} \\ &\quad -x^{n+1}(4n+10x-46nx^2-16nx^3+10nx^4-nx^5-15nx-8x^2+2x^3+4)W^2_{-n} \\ &\quad +4x^{n+1}(n-5nx^2+nx^3-5nx+5x^2-2x^3+1)W_{-n+1}W_{-n}+x(x^4-2x^3+10x^2-2x+1)W_1^2 \\ &\quad +x\left(2x^3-8x^2+10x+4\right)W_0^2-4x\left(-2x^3+5x^2+1\right)W_1W_0. \end{aligned}$$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\sum_{k=1}^{n} k(-1)^k W^2_{-k} = \left.\frac{\frac{d}{dx}(h_3(x))}{\frac{d}{dx}((x+1)^2(x^2-6x+1)^2)}\right|_{x=-1}$$
$$= \frac{1}{32}((-1)^n((-4n^2+27)W^2_{-n+1}+(4n^2+24n-23)W^2_{-n}$$
$$+(8n^2-8n-54)W_{-n+1}W_{-n})-27W_1^2+23W_0^2+54W_1W_0).$$

(b) We use Theorem 4.1 (b). If we set $r=2, s=1$ in Theorem 4.1 (b) then we have

$$\sum_{k=1}^{n} kx^k W_{-k+1}W_{-k} = \frac{h_4(x)}{(x+1)^2(x^2-6x+1)^2}$$

where

$$\begin{aligned} h_4(x) &= -2x^{n+2}(5x-n+5nx^2-nx^3+5nx+x^3-2)W^2_{-n+1} \\ &\quad +2x^{n+1}(n-5nx^2+nx^3-5nx+5x^2-2x^3+1)W^2_{-n} \\ &\quad -x^{n+1}(n+8x-26nx^2-14nx^3+9nx^4-nx^5-nx-18x^2+8x^3+x^4+1)W_{-n+1}W_{-n} \\ &\quad +2x^2(x^3+5x-2)W_1^2-2x\left(-2x^3+5x^2+1\right)W_0^2+x\left(x^4+8x^3-18x^2+8x+1\right)W_1W_0. \end{aligned}$$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned} \sum_{k=1}^{n} k(-1)^k W_{-k+1}W_{-k} &= \left.\frac{\frac{d}{dx}(h_4(x))}{\frac{d}{dx}((x+1)^2(x^2-6x+1)^2)}\right|_{x=-1} \\ &= \frac{1}{32}((-1)^n((-4n^2-4n+27)W^2_{-n+1}+(4n^2-4n-27)W^2_{-n} \\ &\quad +(8n^2+16n-50)W_{-n+1}W_{-n})-27W_1^2+27W_0^2+50W_1W_0). \end{aligned}$$

From the last theorem, we have the following corollary which gives sum formulas of Pell numbers (take $W_n=P_n$ with $P_0=0, P_1=1$).

Corollary 4.9. *For $n\geq 1$, Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(-1)^k P^2_{-k} = \frac{1}{32}((-1)^n((-4n^2+27)P^2_{-n+1}+(4n^2+24n-23)P^2_{-n}+(8n^2-8n-54)P_{-n+1}P_{-n})-27)$.

(b) $\sum_{k=1}^{n} k(-1)^k P_{-k+1}P_{-k} = \frac{1}{32}((-1)^n((-4n^2+27)P^2_{-n+1}+(4n^2+24n-23)P^2_{-n}+(8n^2-8n-54)P_{-n+1}P_{-n})-27)$.

Taking $W_n=Q_n$ with $Q_0=2, Q_1=2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 4.10. *For $n\geq 1$, Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(-1)^k Q^2_{-k} = \frac{1}{32}((-1)^n((-4n^2+27)Q^2_{-n+1}+(4n^2+24n-23)Q^2_{-n}+(8n^2-8n-54)Q_{-n+1}Q_{-n})+200)$.

(b) $\sum_{k=1}^{n} k(-1)^k Q_{-k+1}Q_{-k} = \frac{1}{32}((-1)^n((-4n^2-4n+27)Q^2_{-n+1}+(4n^2-4n-27)Q^2_{-n}+(8n^2+16n-50)Q_{-n}Q_{-n+1})+200)$.

Taking $x=-1, r=1, s=2$ in Theorem 4.1 (a) and (b), we obtain the following proposition.

Proposition 4.1. *If $x=-1, r=1, s=2$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^{k}W_{-k}^{2}=\frac{1}{100}((-1)^{n}((30n+49)W_{-n+1}^{2}-(20n+76)W_{-n}^{2}-4(10n+13)W_{-n+1}W_{-n})-49W_{1}^{2}+76W_{0}^{2}+52W_{1}W_{0})$.

(b) $\sum_{k=1}^{n} k(-1)^{k}W_{-k+1}W_{-k}=\frac{1}{100}((-1)^{n}((10n+23)W_{-n+1}^{2}-4(10n+13)W_{-n}^{2}+(20n-4)W_{-n+1}W_{-n})-23W_{1}^{2}+52W_{0}^{2}+4W_{1}W_{0})$.

From the last proposition, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1$).

Corollary 4.11. *For $n\geq 1$, Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^{k}J_{-k}^{2}=\frac{1}{100}((-1)^{n}((30n+49)J_{-n+1}^{2}-(20n+76)J_{-n}^{2}-4(10n+13)J_{-n+1}J_{-n})-49)$.

(b) $\sum_{k=1}^{n} k(-1)^{k}J_{-k+1}J_{-k}=\frac{1}{100}((-1)^{n}((10n+23)J_{-n+1}^{2}-4(10n+13)J_{-n}^{2}+(20n-4)J_{-n+1}J_{-n})-23)$.

Taking $W_n=j_n$ with $j_0=2, j_1=1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 4.12. *For $n\geq 1$, Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} k(-1)^{k}j_{-k}^{2}=\frac{1}{100}((-1)^{n}((30n+49)j_{-n+1}^{2}-(20n+76)j_{-n}^{2}-4(10n+13)j_{-n+1}j_{-n})+359)$.

(b) $\sum_{k=1}^{n} k(-1)^{k}j_{-k+1}j_{-k}=\frac{1}{100}((-1)^{n}((10n+23)j_{-n+1}^{2}-4(10n+13)j_{-n}^{2}+(20n-4)j_{-n+1}j_{-n})+193)$.

4.3 The Case $x=1+i$

We now consider the complex case $x=1+i$ in Theorem 4.1. The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Taking $x=1+i, r=s=1$ in Theorem 4.1 we obtain the following proposition.

Proposition 4.2. *If $x=1+i, r=s=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} k(-1)^{k}W_{-k}^{2}=\frac{1}{-7+24i}((1+i)^{n}(((7+i)n+3-i)W_{-n+1}^{2}+(i-3-(3-21i)n)W_{-n}^{2}+2((1-7i)n+5+5i)W_{-n+1}W_{-n})-(3-i)W_{1}^{2}+(3-i)W_{0}^{2}-(10+10i)W_{1}W_{0})$.

(b) $\sum_{k=1}^{n} k(-1)^{k}W_{-k+1}W_{-k}=\frac{1}{-7+24i}((1+i)^{n}(((8-6i)n+8+4i)W_{-n+1}^{2}+((1-7i)n+5+5i)W_{-n}^{2}+((5+15i)n+5+5i)W_{-n+1}W_{-n})-(8+4i)W_{1}^{2}-(5+5i)W_{0}^{2}-(5+5i)W_{1}W_{0})$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n=F_n$ with $F_0=0, F_1=1$).

Corollary 4.13. *For $n\geq 1$, Fibonacci numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(1+i)^{k}F_{-k}^{2}=\frac{1}{-7+24i}((1+i)^{n}(((7+i)n+3-i)F_{-n+1}^{2}+(i-3-(3-21i)n)F_{-n}^{2}+2((1-7i)n+5+5i)F_{-n+1}F_{-n})-3+i)$.

(b) $\sum_{k=1}^{n} k(1+i)^{k}F_{-k+1}F_{-k}=\frac{1}{-7+24i}((1+i)^{n}(((8-6i)n+8+4i)F_{-n+1}^{2}+((1-7i)n+5+5i)F_{-n}^{2}+((5+15i)n+5+5i)F_{-n+1}F_{-n})-8-4i)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 4.14. *For $n \geq 1$, Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} k(1+i)^k L_{-k}^2 = \frac{1}{-7+24i}((1+i)^n(((7+i)n+3-i)L_{-n+1}^2 + (i-3-(3-21i)n)L_{-n}^2 + 2((1-7i)n+5+5i)L_{-n+1}L_{-n}) - 3 + i)$.

(b) $\sum_{k=1}^{n} k(1+i)^k L_{-k+1}L_{-k} = \frac{1}{-7+24i}((1+i)^n(((8-6i)n+8+4i)L_{-n+1}^2 + ((1-7i)n+5+5i)L_{-n}^2 + ((5+15i)n+5+5i)L_{-n+1}L_{-n}) - 8 - 4i)$.

Corresponding sums of the other second order linear sequences can be calculated similarly when $x = 1 + i$.

5 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the squares of the numbers of the sequences. Our future work will be investigation of the closed forms of the sum formulas for the squares of generalized Tetranacci numbers.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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