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Integrability of Very Weak Solutions for Boundary Value Problems of Nonhomogeneous p-Harmonic Equations

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Authors' contributions

This work was carried out in collaboration among all authors. Author YZ designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Authors YZ and YT managed the analyses of the study. Author YT managed the literature searches. All authors read and approved the final manuscript.

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Abstract

The paper deals with very weak solutions u to boundary value problems of the nonhomogeneous p-harmonic equation. We show that, any very weak solution u to the boundary value problem is integrable provided that r is sufficiently close to p.

Keywords: Integrability; very weak solution; boundary value problem; p-harmonic equation.

1 Introduction

Let $1 < p < n$, $\theta : \overline{\Omega} \to \mathbf{R}$, $\theta(x) \in W^{1,q}(\Omega)$, $q > r$, $f(x) \in L^{\frac{nq}{n(p-1)+r}}(\Omega)$. We shall examine the boundary value problem of the p-harmonic equation

$$\begin{cases} -\operatorname{div}(|\nabla u(x)|^{p-2} \nabla u(x)) = f(x), & x \in \Omega, \\ u(x) = \theta(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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Throughout this paper Ω will stand for a bounded regular domain in $\mathbf{R}^n (n \geq 2)$. By a regular domain we understand any domain of finite measure for which the estimates (3.3) and (3.4) below for the Hodge decomposition are satisfied, see [1,2]. A Lipschitz domain, for example, is regular.

Definition 1.1. A function $u \in \theta + W_0^{1,r}(\Omega)$, $\max\{1, p-1\} < r < p$, is called a very weak solution to the boundary value problem (1.1) if for all $\Phi \in W_0^{1,r/(r-p+1)}(\Omega)$ with compact support sets in Ω, there is

$$\int_\Omega \left\langle |\nabla u|^{p-2} \nabla u, \nabla \Phi \right\rangle dx = \int_\Omega f(x) \Phi dx \tag{1.2}$$

where $f(x) \in L^{\frac{nq}{n(p-1)+r}}(\Omega)$.

Recall that a function $u \in \theta + W_0^{1,p}(\Omega)$ is called the weak solution of the boundary value problem (1.1) if (1.2) holds true for all $\Phi \in W_0^{1,p}(\Omega)$. The words very weak in Definition 1.1 mean that the Sobolev integrable exponent r of u can be smaller than the natural one p, see [1], Theorem 1, page 602.

In this paper we will need the definition of weak L^t-space (see [2]): for $t > 0$, the weak L^t- space, $L^t_{weak}(\Omega)$, consists of all measurable functions f such that

$$\left| \{ x \in \Omega : |f(x)| > s \} \right| \leq \frac{k}{s^t}$$

for some positive constant $k = k(f)$ and every $s > 0$, where $|E|$ is the n-dimensional Lebesgue measure of E.

Integrability property is important in the regularity theories of nonlinear elliptic PDEs and systems. In [3], Zhu et al. studied the global integrability of nonhomogeneous quasilinear elliptic equations $-\mathrm{div} A(x, u, \nabla u) = f(x) + \mathrm{div}(|\nabla u|^{p-2} \nabla u)$. In [4], Guo et al. studied the higher order integrability of the divergence elliptic equation $-\mathrm{div} A(x, \nabla u) = -\mathrm{div} f$. In [5], Zhang et al. studied the global integrability of A-harmonic equation $-\mathrm{div} A(x, \nabla u) = -\mathrm{div} f$. In this paper, we consider the global integrability of the very weak solutions of the boundary value problem (1.1). The main result is the following theorem.

Theoerm 1.1. Let $\theta \in W^{1,q}(\Omega)$, $q > r$, There exists $\varepsilon_0 = \varepsilon_0(n, p) > 0$, such that for each very weak solution $u \in \theta + W_0^{1,r}(\Omega)$, $\max\{1, p-1\} < r < p < n$, to the boundary value problem (1.1), we have

$$u \in \begin{cases} \theta + L^{q^*}_{weak}(\Omega) & \text{for } q < r, \\ \theta + L^{\tau}_{weak}(\Omega) & \text{for } q = r \text{ and } \tau < \infty, \\ \theta + L^{\infty}(\Omega) & \text{for } q > n, \end{cases} \tag{1.3}$$

provided that $|p - r| < \varepsilon_0$.

Note that we have restricted ourselves to the case $r < n$ since otherwise any function in $W^{1,r}(\Omega)$ is in the spce $L^t(\Omega)$ for any $t < \infty$ by the Sobolev embedding theorem. At the same time, it is also noted that the very weak solution u to the boundary value problem (1.1) is taken from the Sobolev space $W^{1,r}(\Omega)$, and the embedding theorem ensures that the integrability of u reaches from r to r^*. And our result theorem 1.1 improves this integrability. Note that the key to proving the theorem 1.1 is to use Hodge decomposition [1][6] to construct the appropriate test function.

Preliminary Lemmas

Lemma 1.1[6] For $p \ge 2$ and any $X, Y \in \mathbf{R}^n$, one has

$$2^{2-p} |X-Y|^p \le \left\langle |X|^{p-2} X - |Y|^{p-2} Y, X-Y \right\rangle.$$

Here $|\cdot|$ is the Euclidian norm in $\mathbf{R}^n$, and $\langle \cdot, \cdot \rangle$ is the euclidian scalar product.

Lemma 1.2[7] For any $X, Y \in \mathbf{R}^n$, one has

$$\left| |X|^\varepsilon X - |Y|^\varepsilon Y \right| \le \begin{cases} (1+\varepsilon)(|Y|+|X-Y|)^\varepsilon |X-Y|, & \varepsilon > 0, \\ \frac{1-\varepsilon}{2^\varepsilon (1+\varepsilon)} |X-Y|^{1+\varepsilon}, & -1 < \varepsilon \le 0. \end{cases}$$

Lemma 1.3[2] For $1 < p < 2$ and any $X, Y \in \mathbf{R}^n$, one has

$$\left\langle |X|^{p-2} X - |Y|^{p-2} Y, X-Y \right\rangle \ge |X-Y| \left((|X-Y|+|Y|)^{p-1} - |Y|^{p-1} \right).$$

Lemma 1.4[2] Let $\varepsilon_0 > 0$, $\phi : (s_0, \infty) \to [0, \infty)$ is a decrement function such that for each r, s $(r > s > s_0)$, if

$$\phi(r) \le \frac{c}{(r-s)^\alpha} (\phi(s))^\beta$$

where c, α, β are constants, we have

(1) if $\beta > 1$ we have that $\phi(s_0 + d) = 0$, where $d^\alpha = c 2^{\alpha\beta/(\beta-1)} (\phi(s_0))^{\beta-1}$;
(2) If $\beta < 1$ we have that $\phi(s) \le 2^{\mu/(1-\beta)} (c^{1/(1-\beta)} + (2s_0)^\mu \phi(s_0)) s^{-\mu}$, where $\mu = \alpha/(1-\beta)$.

PROOF OF THEOREM 1.1

For any $L > 0$, let

$$v = \begin{cases} u - \theta + L & for\ u - \theta < -L, \\ 0 & for\ -L \le u - \theta \le L, \\ u - \theta - L & for\ u - \theta > L. \end{cases} \tag{3.1}$$

Then according to the hypothesis, we have $v \in W_0^{1,r}(\Omega)$ and $\nabla v = (\nabla u - \nabla \theta) \cdot 1_{\{|u-\theta|>L\}}$, where 1_E is the characteristic function of the set E. We introduce the Hodge decomposition of vector field $|\nabla v|^{p-2} \nabla v \in L^{r/(r-p+1)}(\Omega)$. So that

$$|\nabla v|^{r-p} \nabla v = \nabla \Phi + h. \tag{3.2}$$

Here $\Phi \in W_0^{1, r/(r-p+1)}(\Omega)$, $h \in L^{r/(r-p+1)}(\Omega, \mathbf{R}^n)$ is a vector field with zero divergence, and satisfied

$$\| \nabla \Phi \|_{r/(r-p+1)} \le C(n,p) \| \nabla v \|_r^{r-p+1} \tag{3.3}$$

and

$$\| h \|_{r/(r-p+1)} \le C(n,p) | p-r | \| \nabla v \|_r^{r-p+1} . \tag{3.4}$$

From the counter-proof method, it is inevitable to exist φ such that $\Phi = \varphi - \varphi_\Omega$. Taken Φ as a test function of the integral identity (1.2), that is

$$\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u, |\nabla u - \nabla \theta|^{r-p} (\nabla u - \nabla \theta) \right\rangle dx = \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u, h \right\rangle dx + \int_{\{|u-\theta|>L\}} f(x) \Phi dx.$$

This implies

$$\begin{aligned}
&\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u - |\nabla \theta|^{p-2} \nabla \theta, |\nabla u - \nabla \theta|^{r-p} (\nabla u - \nabla \theta) \right\rangle dx \\
&= \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u - |\nabla \theta|^{p-2} \nabla \theta, h \right\rangle dx \\
&\quad + \int_{\{|u-\theta|>L\}} \left\langle |\nabla \theta|^{p-2} \nabla \theta, h \right\rangle dx \\
&\quad - \int_{\{|u-\theta|>L\}} \left\langle |\nabla \theta|^{p-2} \nabla \theta, |\nabla u - \nabla \theta|^{r-p} (\nabla u - \nabla \theta) \right\rangle dx \\
&\quad + \int_{\{|u-\theta|>L\}} f(x) \Phi dx \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.5}$$

Now we shall distinguish between two cases.

Case 1: $p \ge 2$. Using Lemma 2.1, (3.5) can be estimated as

$$\begin{aligned}
&\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u - |\nabla \theta|^{p-2} \nabla \theta, |\nabla u - \nabla \theta|^{r-p} (\nabla u - \nabla \theta) \right\rangle dx \\
&\ge 2^{2-p} \int_{\{|u-\theta|>L\}} |\nabla u - \nabla \theta|^r \, dx.
\end{aligned} \tag{3.6}$$

Using the Lemma 2.2, Hölder inequality and Young inequality, $|I_1|$ can be estimated as

$$\begin{aligned}
|I_1| &= \left| \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u - |\nabla \theta|^{p-2} \nabla \theta, h \right\rangle dx \right| \\
&\le (p-1) \int_{\{|u-\theta|>L\}} (|\nabla \theta| + |\nabla u - \nabla \theta|)^{p-2} |\nabla u - \nabla \theta| |h| \, dx \\
&\le 2^{p-2}(p-1) \left(\int_{\{|u-\theta|>L\}} |\nabla \theta|^{p-2} |\nabla u - \nabla \theta| |h| \, dx + \int_{\{|u-\theta|>L\}} |\nabla u - \nabla \theta|^{p-1} |h| \, dx \right) \\
&\le 2^{p-2}(p-1) \Big[\Big(\int_{\{|u-\theta|>L\}} |\nabla \theta|^r \, dx \Big)^{\frac{p-2}{r}} \Big(\int_{\{|u-\theta|>L\}} |\nabla u - \nabla \theta|^r \, dx \Big)^{\frac{1}{r}} \\
&\quad \cdot \Big(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}} \, dx \Big)^{\frac{r-p+1}{r}} + \Big(\int_{\{|u-\theta|>L\}} |\nabla u - \nabla \theta|^r \, dx \Big)^{\frac{p-1}{r}} \cdot \Big(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}} \, dx \Big)^{\frac{r-p+1}{r}} \Big] \\
&\le 2^{p-2}(p-1) C(n,p) |p-r| \Big[\Big(\int_{\{|u-\theta|>L\}} |\nabla \theta|^r \, dx \Big)^{\frac{p-2}{r}} \\
&\quad \cdot \Big(\int_{\{|u-\theta|>L\}} |\nabla u - \nabla \theta|^r \, dx \Big)^{\frac{r-p+2}{r}} + \int_{\{|u-\theta|>L\}} |\nabla u - \nabla \theta|^r \, dx \Big].
\end{aligned} \tag{3.7}$$

Using the Hölder inequality, (3.4) and Young inequality, $|I_2|$ and $|I_3|$ can be estimated as

$$
\begin{aligned}
|I_2| &= \left|\int_{\{|u-\theta|>L\}} \langle |\nabla\theta|^{p-2}\nabla\theta, h\rangle dx\right| \\
&\le \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1}|h|\, dx \\
&\le \left(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\, dx\right)^{\frac{p-1}{r}} \left(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}}\, dx\right)^{\frac{r-p+1}{r}} \\
&\le C(n,p)|p-r| \left(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\, dx\right)^{\frac{p-1}{r}} \cdot \left(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx\right)^{\frac{r-p+1}{r}} \\
&\le C(n,p)|p-r| \left[C(\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\, dx + \varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx\right],
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
|I_3| &= \left|-\int_{\{|u-\theta|>L\}} \langle |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\rangle dx\right| \\
&\le \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1}|\nabla u-\nabla\theta|^{r-p+1}\, dx \\
&\le \left(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\, dx\right)^{\frac{p-1}{r}} \left(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx\right)^{\frac{r-p+1}{r}} \\
&\le C(\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\, dx + \varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx.
\end{aligned}
\tag{3.9}
$$

Using the Hölder inequality, Sobolev-Poincáre inequality[8],

$$
\left(\int_{\Omega} |u-u_{\Omega}|^{pn/(n-p)} dx\right)^{(n-p)/pn} \le C\left(\int_{\Omega} |\nabla u|^{p} dx\right)^{1/p}, (1\le p<n),
$$

and using (3.3) and Young inequality, $|I_4|$ can be estimated as

$$
\begin{aligned}
|I_4| &= \left|\int_{\{|u-\theta|>L\}} f(x)\Phi dx\right| \\
&\le \left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{nr}} \cdot \left(\int_{\{|u-\theta|>L\}} |\varphi-\varphi_{\Omega}|^{\frac{nr}{n(r-p+1)-r}}\, dx\right)^{\frac{n(r-p+1)-r}{nr}} \\
&\le C(n,p)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{nr}} \cdot \left(\int_{\{|u-\theta|>L\}} |\nabla\Phi|^{\frac{r}{r-p+1}}\, dx\right)^{\frac{r-p+1}{r}} \\
&\le C(n,p)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{nr}} \cdot \left(\int_{\{|u-\theta|>L\}} |\nabla v|^{r}\, dx\right)^{\frac{r-p+1}{r}} \\
&\le C(n,p)\left[C(\varepsilon)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{n(p-1)}} \right. \\
&\quad \left. +\varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx\right].
\end{aligned}
\tag{3.10}
$$

Combining (3.5)-(3.10), we arrive at

$$
\begin{aligned}
&\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx \\
&\le C(n,p,\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\, dx \\
&\quad +(C(n,p)|p-r|+\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx \\
&\quad +C(n,p,\varepsilon)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{n(p-1)}},
\end{aligned}
\tag{3.11}
$$

Case 2: $1<p<2$. Lemma 2.3 yields

$$\begin{aligned}&\int_{\{|u-\theta|>L\}}\left\langle |\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx\\ &\geq\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r-p+1}\\ &\qquad\cdot((|\nabla u-\nabla\theta|+|\nabla\theta|)^{p-1}-|\nabla\theta|^{p-1})dx.\end{aligned}$$

This implies

$$\begin{aligned}&\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r}\,dx\\ &\leq\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r-p+1}(|\nabla u-\nabla\theta|+|\nabla\theta|)^{p-1}dx\\ &\leq\int_{\{|u-\theta|>L\}}\left\langle |\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx\\ &\quad+\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r-p+1}|\nabla\theta|^{p-1}\,dx\\ &\leq\int_{\{|u-\theta|>L\}}\left\langle |\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx\\ &\quad+\varepsilon\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r}\,dx+C(\varepsilon)\int_{\{|u-\theta|>L\}}|\nabla\theta|^{r}\,dx.\end{aligned}\tag{3.12}$$

Using Lemma 2.2 and (3.4), $|I_1|$ can be estimated as

$$\begin{aligned}|I_1|&=\left|\int_{\{|u-\theta|>L\}}\left\langle |\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta, h\right\rangle dx\right|\\ &\leq\frac{3-p}{2^{p-2}(p-1)}\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{p-1}|h|\,dx\\ &\leq\frac{3-p}{2^{p-2}(p-1)}\left(\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r}\,dx\right)^{\frac{p-1}{r}}\cdot\left(\int_{\{|u-\theta|>L\}}|h|^{\frac{r}{r-p+1}}\,dx\right)^{\frac{r-p+1}{r}}\\ &\leq\frac{3-p}{2^{p-2}(p-1)}C(n,p)|p-r|\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r}\,dx.\end{aligned}\tag{3.13}$$

For the case $1<p<2$, $|I_2|$ and $|I_3|$ can also be estimated by (3.8)-(3.9). Combining (3.5), (3.12) and (3.13), we arrive at (3.11).

Let $\varepsilon_0=1/C(n,p)$. Then for $|p-r|<\varepsilon_0$ we have $C(n,p)|p-r|<1$. Taking ε small enough, such that $C(n,p)|p-r|+\varepsilon<1$, then the second term on the right-hand side of (3.11) can be absorbed by the left-hand side; thus we obtain

$$\begin{aligned}&\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r}\,dx\\ &\leq C(n,p)\int_{\{|u-\theta|>L\}}|\nabla\theta|^{r}\,dx+C(n,p)\left(\int_{\{|u-\theta|>L\}}|f(x)|^{\frac{nr}{n(p-1)+r}}\,dx\right)^{\frac{n(p-1)+r}{n(p-1)}}.\end{aligned}\tag{3.14}$$

Since $\theta\in W^{1,q}(\Omega)$, $q>r$, using the Hölder inequality, we have

$$\begin{aligned}&\int_{\{|u-\theta|>L\}}|\nabla\theta|^{r}\,dx\\ &\leq\left(\int_{\{|u-\theta|>L\}}|\nabla\theta|^{q}\,dx\right)^{r/q}|\{|u-\theta|>L\}|^{(q-r)/q}\\ &=\|\nabla\theta\|_q^r|\{|u-\theta|>L\}|^{(q-r)/q}.\end{aligned}\tag{3.15}$$

By the proof idea of reference [9](Page 442), and the Hölder inequality, we get

$$\begin{aligned}&(\int_{\{|u-\theta|>L\}}|f(x)|^{\frac{nr}{n(p-1)+r}}dx)^{\frac{n(p-1)+r}{n(p-1)}}\\&\le(\int_{\{|u-\theta|>L\}}|f(x)|^{\frac{nq}{n(p-1)+r}}dx)^{\frac{nr(p-1)+r^2}{qn(p-1)}}|\{|u-\theta|>L\}|^{(q-r)/q}\\&\le M|\{|u-\theta|>L\}|^{(q-r)/q},\end{aligned}\tag{3.16}$$

where $M=(\int_{\{|u-\theta|>L\}}|f(x)|^{\frac{nq}{n(p-1)+r}}dx)^{\frac{nr(p-1)+r^2}{qn(p-1)}}$, M is bounded and is a constant dependent only on n, p. Then (3.14) can be collated into the following results

$$\begin{aligned}&\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r dx\\&\le C(n,p)(\int_{\{|u-\theta|>L\}}|\nabla\theta|^q dx)^{r/q}|\{|u-\theta|>L\}|^{(q-r)/q}\\&\quad+C(n,p)M|\{|u-\theta|>L\}|^{(q-r)/q}\\&=C|\{|u-\theta|>L\}|^{(q-r)/q}(1+\|\nabla\theta\|_q^r)\ ,\end{aligned}\tag{3.17}$$

where $C=C(n,p,M)$.

We now turn our attention back to the function $v\in W_0^{1,r}(\Omega)$. By the Sobolev embedding theorem, we have

$$\begin{aligned}(\int_\Omega|v|^{r^*}dx)^{1/r^*}&\le C(n,r)(\int_\Omega|\nabla v|^r dx)^{1/r}\\&=C(n,r)(\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r dx)^{1/r},\end{aligned}\tag{3.18}$$

since $|v|=(|u-\theta|-L)\cdot 1_{\{|u-\theta|>L\}}$, we have

$$(\int_{\{|u-\theta|>L\}}(|\nabla u-\nabla\theta|-L)^{r^*}dx)^{1/r^*}=(\int_\Omega|v|^{r^*}dx)^{1/r^*},\tag{3.19}$$

and for $L'>L$,

$$\begin{aligned}&(L'-L)^{r^*}|\{|u-\theta|>L'\}|\\&=\int_{\{|u-\theta|>L\}}(L'-L)^{r^*}dx\\&\le\int_{\{|u-\theta|>L'\}}(|u-\theta|-L)^{r^*}dx\\&\le\int_{\{|u-\theta|>L\}}(|u-\theta|-L)^{r^*}dx.\end{aligned}\tag{3.20}$$

By collecting (3.17)-(3.20), we deduce that

$$\begin{aligned}&((L'-L)^{r^*}|\{|u-\theta|>L'\}|)^{1/r^*}\\&\le C(n,r)(\|\nabla\theta\|_q+1)|\{|u-\theta|>L\}|^{1/r-1/q}.\end{aligned}\tag{3.21}$$

Thus

$$\begin{aligned}&|\{|u-\theta|>L'\}|\\&\le\frac{1}{(L'-L)^{r^*}}(C(n,r)(\|\nabla\theta\|_q+1))^{r^*}|\{|u-\theta|>L\}|^{r^*(1/r-1/q)}.\end{aligned}\tag{3.22}$$

Let $\phi(s)=|\{|u-\theta|>s\}|$, $\alpha=r*$, $c=(C(n,r)(\|\nabla\theta\|_q+1))^{r^*}$, $\beta=r*(1/r-1/q)$, $s_0>0$, Then (3.22) become

$$\phi(L')\le\frac{c}{(L'-L)^{\alpha}}\phi(L)^{\beta} \tag{3.23}$$

for $L'>L>0$.

(1) For the case $q<n$, one has $\beta<1$. In this case, if $s\ge1$, we get from Lemma 2.3 that

$$|\{|u-\theta|>s\}|\le c(\alpha,\beta,s_0)s^{-t},$$

where $t=\alpha/(1-\beta)=q*$. For $0<s<1$, one has

$$|\{|u-\theta|>s\}|\le|\Omega|=|\Omega|s^{q^*}s^{-q^*}\le|\Omega|s^{-q^*}.$$

Thus

$$u\in\theta+L_{weak}^{q^*}(\Omega).$$

(2) For the case $q=n$, one has $\beta=1$. For any $\tau<\infty$, (3.23) implies

$$\begin{aligned}\phi(L')&\le\frac{c}{(L'-L)^{\alpha}}\phi(L)=\frac{c}{(L'-L)^{\alpha}}\phi(L)^{1-\alpha/\tau}\phi(L)^{\alpha/\tau}\\&\le\frac{c|\Omega|^{\alpha/\tau}}{(L'-L)^{\alpha}}\phi(L)^{1-\alpha/\tau}.\end{aligned}$$

As about, we derive

$$u\in\theta+L_{weak}^{\tau}(\Omega).$$

(3) For the case $q>n$, one has $\beta>1$. Lemma 2.3 implies $\phi(d)=0$ for some $d=d(\alpha,\beta,s_0,r,(\|\nabla\theta\|_q+1))$. Thus $|\{|u-\theta|>d\}|=0$, which means $u-\theta\le d$ a.e. in Ω. Therefore

$$u\in\theta+L^{\infty}(\Omega),$$

completing the proof of Theorem 1.1.

Competing Interests

Authors have declared that no competing interests exist.

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