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Nonlinear Elliptic Problems Involving the $(p(x), q(x))$-Laplacian System in $\mathbb{R}^n$

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Short Research Article

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Abstract

In this article, we study the elliptic problems involving $(p(x), q(x))$-Laplacian in $\mathbb{R}^n$, We apply a result by [1] to prove the existence of multiple nontrivial solutions.

Keywords: p(x)-laplacian; (p(x),q(x))-laplacian; maximal growth; sobolev inequality; ricceri's principle.

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1 Introduction

In this article, we study the following two nonlinear problems:

$$\begin{gathered} -\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\,\nabla u) + b(x)|u|^{p(x)-2}u = \lambda\, f(x,u) + \mu\, g(x,u) \quad \text{in } \mathbb{R}^N \\ u \in W_0^{1,p(x)}(\mathbb{R}^N) \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} -\operatorname{div}\left(a(x)\nabla u|^{p(x)-2}\nabla u\right) + b(x)|u|^{p(x)-2}u = \lambda f_u(x,u,v) + \mu g_u(x,u,v) \quad \text{in } \mathbb{R}^N \\ -\operatorname{div}\left(a'(x)|\nabla v|^{q(x)-2}\nabla v\right) + b'(x)|u|^{q(x)-2}u = \lambda f_v(x,u,v) + \mu g_v(x,u,v) \quad \text{in } \mathbb{R}^N \\ u \in W_0^{1,p(x)}(\mathbb{R}^N), v \in W_0^{1,q(x)}(\mathbb{R}^N) \end{gathered} \tag{1.2}$$

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For the one equation problem (1.1), we assume that $3 \leq N; p : \mathbb{R}^N \longrightarrow \mathbb{R}$ is Lipchitz continuous with $2 \leq p^- \leq p(x) \leq p^+ < N$. λ and μ are positives parameters, and a is a measurable function such that $a \in L^\infty(\mathbb{R}^n)$ with $\inf_{x\in\mathbb{R}^n} a(x) > 0$. The two functions $f(x,t)$ and $g(x,t)$ having subcritical growth with respect to t, more precisely, we assume that f is a Carathéodory function satisfying the following condition

$$|f(x,t)| \leq m(x)|t|^\gamma \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall t \in \mathbb{R}, \tag{1.3}$$

where m is a positive function such that $m \in L^{\frac{p^*_-}{p^*_- -1}}(\mathbb{R}^n) \cap L^{\frac{\nu}{\nu-1}(\frac{p^*_-}{p^*_- -(\gamma+1)})}(\mathbb{R}^n)$, with $p^+ < \gamma+1 < \nu < p^*_-, p^*_-$ denotes the critical Sobolev exponent, i.e., $p^*_- = \dfrac{np^-}{n-p^+}$.

The function $g = g(x,t)$, need to be a measurable function with respect to x in $\mathbb{R}^n$ for every t in $\mathbb{R}$, and is continuous with respect to t in $\mathbb{R}$ for almost every x in $\mathbb{R}^n$. Such that $g(x,0) = 0$, and there exists a positive function $h(x) \in L^\infty(\mathbb{R}^n)$, satisfying

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}-\{0\}} \frac{|g(x,t)|}{h(x)|t|^{p^*_-}} < +\infty \tag{1.4}$$

The function $b(x)$ needs to satisfy the next condition:

$$b(x) > b_0 > 0 \ \forall x \in \mathbb{R}^N \tag{1.5}$$

Problems involving p(x)-growth conditions, such as (1.1), have been given a special attention since they are important in modelling some physical phenomena. For example their applications to the study of electrorheological fluids and in elastic mechanics (see [2], [3], [4], [5]). The case of the degenerate equation $-\text{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x,u)$ in $\mathbb{R}^n$, was studied by many authors, we mention the work of P.[6], and others like [7], [8], [9]. Existence results for p(x)-Laplacian Dirichlet problems on bounded domains we refer to [[10], [11]] while for the study of p(x)-Laplacian problems in $\mathbb{R}^N$ we refer to [[12], [13], [14], and [15]]. In both investigation i.e bounded and unbounded, the authors used a standard approach by applying the Mountain Pass Lemma for finding critical points of associated variational formulations of Ambrosetti and Rabinowitz [16].

Since the appearance of the abstract result proved by Ricceri in [17]and its revisited note established in [18] dealing with variational equations with both Dirichlet and Neumann conditions, they have extensively been investigated. In [1], Ricceri obtained a general three critical points theorem, that has been applied for a class of elliptic operators involving nonlinearities of polynomial growth. We mention that the result proved in [1], will be an essential tool for the study of the existence of at least three weak solutions for problems (1.1) and (1.2), when the nonlinearities have a maximal growth.

This article is organized as follows. In section 2 we introduce the generalized weighted Lebesgue-Sobolev spaces $L^{p(x)}_{b(x)}(\mathbb{R}^N)$ and $W^{1,p(x),b(x)}(\mathbb{R}^N)$, and some imbedding results. In section 3 we treat the the case of the one elliptic equation involving p(x)-Laplacian (1.1) and we will prove multiple results by applying Ricceri's principle in [18]. Finally, in section 4, we extend the result of the last section to general elliptic systems of two nonlinear partial differential equations governed essentially by the (p(x),q(x))-Laplacian (1.2)). Let us first recall the crucial theorem

Theorem 1.1. [1] Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$, a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X, and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum x_0 with $\Phi(x_0) = J(x_0) = 0$.

Finally, set

$$\alpha = \max\{0, \limsup_{\|x\|\to+\infty} \frac{J(x)}{\Phi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Phi(x)}\},$$

$$\beta = \sup_{x\in\Phi^{-1}(]0,+\infty[)} \frac{J(x)}{\Phi(x)},$$

and assume that $\alpha < \beta$. Then, for each compact interval $[a,b] \subset]1/\beta, 1/\alpha[$ (with the conventions $1/0 = +\infty$, $1/\infty = 0$) there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$, and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\sigma > 0$ such that for each $\mu \in [0,\sigma]$, the equation

$$\Phi'(x) = \lambda J'(x) + \mu\Psi'(x)$$

has at least three solutions in X whose norms are less than r.

2 Abstract Framework

for a $b(x) \in L_{loc}(\mathbb{R}^N)$ satisfying the condition (1.5) we introduce the generalized weighted Lebesgue-Sobolev spaces $L^{p(x)}_{b(x)}(\mathbb{R}^N)$ and $W^{1,p(x),b(x)}(\mathbb{R}^N)$

$$L^{p(x)}_{b(x)}(\mathbb{R}^N) = \{u, \int_{\mathbb{R}^N} b(x)|u(x)|^{p(x)} < +\infty\}$$

equipped with the so called Luxembourg norm

$$\|u\|_{b(x),p(x)} = inf\{\nu > 0, \int_{\mathbb{R}^N} b(x)|\frac{u(x)}{\nu}|^{p(x)}dx < 1\}$$

and

$$W^{1,p(x),b(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}_{b(x)}(\mathbb{R}^N), \nabla u \in L^{p(x)}(\mathbb{R}^N)\}$$

endowed with the norm

$$\|u\|_{1,p(x),b(x)} = \|u\|_{p(x),b(x)} + \|\nabla u\|_{a(x),p(x)}$$

variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respect, they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- < p^+ < \infty$ with $p^- = \inf_{x\in\mathbb{R}^N} p(x)$ and $p^+ = \sup_{x\in\mathbb{R}^N} p(x)$ and continuous functions are dense if $p^+ < \infty$ ([19] Theorem 2.8). we denote by $L^{p'(x)}(\mathbb{R}^N)$ the conjugate space of $L^{p(x)}(\mathbb{R}^N)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ .For any $u \in L^{p(x)}(\mathbb{R}^N), v \in L^{p'(x)}(\mathbb{R}^N)$ The Hölde type inequality

$$|\int_{\mathbb{R}^N} uvdx| \le (\frac{1}{p^-} + \frac{1}{p'^-})\|u\|_{L^{p(x)}}\|v\|_{L^{p'(x)}} \tag{2.1}$$

holds true.

Let E be the space defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{1,p(x),b(x)}$, the condition (1.5) on the function $b(x)$ implies that $E \subset W_0^{1,p}(\mathbb{R}^n)$. Also a simple calculation shows that the norm $\|u\|_{1,p(x),b(x)}$ is equivalent to the norm

$$\|u\| = inf\ \{\nu > 0, \int_{\mathbb{R}^n} |\frac{a(x)\nabla u(x)}{\nu}|^{p(x)}dx + |\frac{b(x)u(x)}{\nu}|^{p(x)}dx < 1\}. \tag{2.2}$$

set

$$I(u) = \int_{\mathbb{R}^n} a(x)|\nabla u(x)|^{p(x)}dx + b(x)|u(x)|^{p(x)}dx$$

then for all $u \in E$

$$\|u\| > 1 \implies \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+} \tag{2.3}$$

$$\|u\| < 1 \implies \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-} \tag{2.4}$$

$u \in E$ is a weak solution of (1.1) if

$$\int_{\mathbb{R}^N} \left(a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v + b(x)|u|^{p(x)-2}uv\right) dx = \int_{\mathbb{R}^N} \lambda f(x,u)v dx + \int_{\mathbb{R}^N} \mu g(x,u)v\, dx$$

$\forall v \in E$.

Let recall some embedding results concerning variable exponent Lebesque-Sobolev spaces. If $p(x)$ is Lipchitz continuous and $p^+ < N$, then for any $q(x)$ satisfying $p(x) \leq q(x) \leq \frac{Np(x)}{N-p(x)}$, the embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}$ is continuous ([20] theorem 1.1). Also, the Sobolev inequality, $\|u\|_{L^{p(x)^*}} \leq C\|u\|$ holds for all $u \in E$ for some constant $C > 0$.

3 Main Results and Proof

We state and prove the following main result of this paper.

Theorem 3.1. *Let us suppose* $2 < p(x) < N$ *and* (1.3) . *Furthermore, suppose for* $1 < \tau < p^-$ *and some positive function* $\alpha(x) \in L^{(\frac{p^*}{\tau})'}(\mathbb{R}^n)$ *that*

$$\limsup_{|u|\to+\infty} \frac{F(x,u)}{\alpha(x)|u|^\tau} \leq M < +\infty \quad \textit{uniformly} \quad x \in \mathbb{R}^N \tag{3.1}$$

and

$$\sup_{u\in E}\{\int_{\mathbb{R}^N} F(x,u)dx\} > 0$$

where $F(x,t) = \int_0^t f(x,s)\, ds$

If we set

$$\omega = \frac{1}{p^+} \inf\left\{\frac{\int_{\mathbb{R}^n} a(x)|\nabla u|^{p(x)} + \int_{\mathbb{R}^n} b(x)|u|^{p(x)}dx}{\int_{\mathbb{R}^n} F(x,u)\, dx} : u \in E, \int_{\mathbb{R}^n} F(x,u)\, dx > 0\right\} \tag{3.2}$$

Then for each compact interval $[a,b] \subset]\omega, +\infty[$, *there exists* $r_1 > 0$ *with the following property: for every* $\lambda \in [a,b]$, *and every function* $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *which is measurable in* $\mathbb{R}^N$ *and continuous in* $\mathbb{R}$ *satisfying* (1.4), *there exists* $\delta > 0$ *such that for each* $\mu \in [0,\delta]$, *the one equation problem* (1.1) *has at least three nonzero weak solutions in* E *whose norms are less than* r_1.

To prove our result we need to define the following functions $\Phi, J, \Psi : E \longrightarrow \mathbb{R}$

$$\Phi(u) = \int_{\mathbb{R}^n} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\right) dx$$

and

$$J(u) = \int_{\mathbb{R}^n} F(x,u)\, dx$$

and

$$\Psi(u) = \int_{\mathbb{R}^n} G(x,u)\, dx$$

the function $\phi(u)$ satisfy the following inequalities

$$\frac{I(u)}{p^+} \leq \phi(u) \leq \frac{I(u)}{p^-}$$

then using (2.3) and (2.4) ,

$$\frac{\|u\|^{p^-}}{p^+} \leq \Phi(u) \leq \frac{\|u\|^{p^+}}{p^-} \quad if \;\; \|u\| \leq 1 \tag{3.3}$$

$$\frac{\|u\|^{p^+}}{p^+} \leq \Phi(u) \leq \frac{\|u\|^{p^-}}{p^-} \quad if \;\; 1 \leq \|u\| \tag{3.4}$$

for each $u \in E$ It is well known that Φ is well defined and continuously Gâteaux differentiable with

$$\Phi^{'}(u)v = \int_{\mathbb{R}^n} a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v + \int_{\mathbb{R}^n} b(x)|u|^{p(x)-2}uv\,dx$$

for all $u, v \in E$. Note also that Φ is clearly coercive, weakly lower semi-continuous and bounded on each bounded subset of E. By the standard uniform convexity algebraic inequality of the function $l(x) = |x|^{p(x)}, x \in \mathbb{R}^n$, we deduce that Φ is uniformly monotone operator in E Moreover, a classical result on uniformly convex spaces ensures that $\Phi^{'}$ admits a continuous inverse on $(E)^*$ (Theorem 26. A) of [21], i.e. $\Phi \in W_E$.
It follows from (1.3),

$$|F(x, u)| \leq \frac{1}{\gamma+1} m(x)|u|^{\gamma+1} \tag{3.5}$$

for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$. Then by standard argument we have F is in $C^1(\mathbb{R}^n \times \mathbb{R})$, hence we see that J is well defined and continuously Gâteaux differentiable with

$$J^{'}(u)v = \int_{\mathbb{R}^n} f(x, u)v\,dx$$

for all $u, v \in E$.

Lemma 3.2. *J' is a compact map from E to $(E)^*$.*

Proof.

Let $\{u_k\}$ be a sequence in E which converges weakly to u. On one hand, in view of Hölder's inequality and Sobolev embedding, we obtain for all $0 \leq R \leq +\infty$,

$$\int_{|x|\geq R} f(x, u)v\,dx \leq \left(\int_{|x|\geq R} |m|^{p_1}\,dx\right)^{\frac{1}{p_1}} (C_1\|u\|)^{\gamma}\|v\|^{p^*_-},$$

for all $u, v \in E$, and $p_1 = \frac{p^*_-}{p^*_- -(\gamma+1)} \in \left[\frac{p^*_-}{p^*_- -1}, \frac{\nu}{\nu-1}(\frac{p^*_-}{P^*_- -(\gamma+1)})\right]$. Since $m \in L^{p_1}(\mathbb{R}^n)$, we have $\lim_{R\to+\infty} \int_{|x|\geq R} |m|^{p_1}\,dx = 0$. This implies with the fact that $\{u_k\}$ is a bounded sequence, for any ε, there exists $R_\varepsilon > 0$ such that

$$\int_{|x|\geq R_\varepsilon} f(x, u_k)v\,dx \leq \varepsilon \quad \text{and} \quad \int_{|x|\geq R_\varepsilon} f(x, u)v\,dx \leq \varepsilon \tag{3.6}$$

holds for all k.

On the other hand, applying Young's inequality, we get

$$\begin{aligned} f^{\frac{\nu}{\nu-1}}(x, t) &\leq m(x)^{\frac{\nu}{\nu-1}} t^{\frac{\gamma\nu}{\nu-1}} \\ &\leq \frac{p^*_- - (\gamma+1)}{p^*_-} m(x)^{\frac{\nu}{\nu-1}(\frac{p^*_-}{p^*_-(\gamma+1)})} + \frac{\gamma+1}{p^*_-} t^{\frac{\gamma\nu}{\nu-1}\frac{p^*_-}{\gamma+1}}, \end{aligned}$$

for all $t \in \mathbb{R}$ and a.e. $x \in B_\varepsilon = \{x \in \mathbb{R}^n; |x| < R_\varepsilon\}$. Let us remark that

$$\frac{\gamma\nu}{\nu - 1}\frac{p_-^*}{\gamma + 1} < p_-^*,$$

since $\gamma + 1 < \nu$. Hence the Nemytskii operator $N_{f^{\frac{\nu}{\nu-1}}}$ associated with $f^{\frac{\nu}{\nu-1}}$ is continuous from $L^{\frac{\gamma\nu}{\nu-1}\frac{p_-^*}{\gamma+1}}(B_\varepsilon)$ in $L^1(B_\varepsilon)$. Then we conclude that

$$\int_{|x|<R_\varepsilon} f(x, u_k)^{\frac{\nu}{\nu-1}}\, dx \to \int_{|x|<R_\varepsilon} f(x, u)^{\frac{\nu}{\nu-1}}\, dx,$$

which implies that $f(x, u_k)$ converges to $f(x, u)$ in $L^{\frac{\nu}{\nu-1}}(B_\varepsilon)$. Hence since $L^{p_-^*}(B_\varepsilon) \subset L^\nu(B_\varepsilon)$, we have $f(x, u_k)v$ converges to $f(x, u)v$ in $L^1(B_\varepsilon)$, i.e.

$$\int_{|x|<R_\varepsilon} (f(x, u_k) - f(x, u))v\, dx \to 0, \tag{3.7}$$

for all $v \in L^\nu(B_\varepsilon)$. Finally, in view of (3.6) and (3.7), we get

$$\int_{\mathbb{R}^n} f(x, u_k)v\, dx \to \int_{\mathbb{R}^n} f(x, u)v\, dx.$$

This completes the proof of Lemma 2.1 and consequently $J^{'}$ is compact.

Now in order to prove our result, we prove that the conditions given in Theorem 1.1 are satisfied. Indeed, using Hölder's inequality and Sobolev embedding, we have in view of (3.5),

$$\int_{\mathbb{R}^n} F(x, u)\, dx \le \frac{1}{\gamma + 1}\left(\int_{\mathbb{R}^n} |m|^{p_1}\, dx\right)^{\frac{1}{p_1}} (C_1\|u\|)^{\gamma+1}$$

$\forall (x, u) \in \mathbb{R}^n \times E$ with $p_1 = \frac{p_-^*}{p_-^* - (\gamma+1)}$.

Then from (3.3) (3.4) we obtain

$$\frac{J(u)}{\Phi(u)} \le \frac{p^+}{\gamma + 1} C_1^{\gamma+1} \|m\|_{L^{p_1}} \frac{\|u\|^{\gamma+1}}{\|u\|^{p^-}}, \quad \|u\| \le 1$$

and

$$\frac{J(u)}{\Phi(u)} \le \frac{p^+}{\gamma + 1} C_1^{\gamma+1} \|m\|_{L^{p_1}} \frac{\|u\|^{\gamma+1}}{\|u\|^{p^+}}, \quad 1 \le \|u\|$$

for all $u \in E$. Hence since $p^- < p^+ < \gamma + 1$ we have for $\varepsilon > 0$ small enough

$$\limsup_{u\to 0} \frac{J(u)}{\Phi(u)} \le \frac{p^+}{\gamma + 1} {C_1}^{\gamma+1} \|m\|_{L^{p_1}} \varepsilon. \tag{3.8}$$

On the other hand, using (3.1), there exists $A > 0$ such that

$$|F(x, u)| \le M\alpha(x)|u|^\tau, \quad \forall |u| > A$$

where $\tau < p^- < p^+$ and $M > 0$. Then, using Hölder and Young inequalities and Sobolev embedding, we obtain for each $u \in E\backslash\{0\}$ for $\|u\| > 1$

$$\begin{aligned}
\frac{J(u)}{\Phi(u)} &\le \frac{p^+ \int_{\mathbb{R}^n(|u|>A)} F(x, u)\, dx}{\|u\|^{p^-}} + \frac{p^+ \int_{\mathbb{R}^n(|u|\le A)} F(x, u)\, dx}{\|u\|^{p^-}} \\
&\le \frac{p^+ M \int_{\mathbb{R}^n(|u|>A)} \alpha(x)|u|^\tau\, dx}{\|u\|^{p^-}} + \frac{p^+}{\gamma + 1} \frac{\int_{\mathbb{R}^n(|u|\le A)} m(x)|u|^{\gamma+1} dx}{\|u\|^{p^-}} \\
&\le \frac{pMC_1^\tau \|\alpha\|_{L^{(\frac{p_-^*}{\tau})'}} \|u\|^\tau}{\|u\|^{p^-}} + \frac{p^+ A^{\gamma+1-\tau}}{\gamma + 1} \frac{\int_{\mathbb{R}^n(|u|\le A)} m(x)|u|^\tau dx}{\|u\|^{p^-}}
\end{aligned}$$

Now using the fact that $m \in L^{\frac{p^*}{p^*_- - \tau}}(\mathbb{R}^n)$ since $\frac{p^*_-}{p^*_- - \tau} \in [\frac{p^*_-}{p^*_- - 1}, \frac{p^*_-}{p^*_- - (\gamma+1)}]$, we get

$$\begin{aligned} \frac{J(u)}{\Phi(u)} &\leq \frac{p^+ M C_1^\tau \|\alpha\|_{L^{(\frac{p^*_-}{\tau})'}}}{\|u\|^{p^- - \tau}} + \frac{p^+ A^{\gamma+1-\tau} C_1^\tau}{\gamma+1} \frac{\|m\|_{L^{\frac{p^*_-}{p^*_- - \tau}}} \|u\|^\tau}{\|u\|^{p^-}} \\ &\leq \frac{p^+ M C_1^\tau \|\alpha\|_{L^{(\frac{p^*_-}{\tau})'}}}{\|u\|^{p^- - \tau}} + \frac{p^+ A^{\gamma+1-\tau} C_1^\tau}{\gamma+1} \frac{\|m\|_{L^{\frac{p^*_-}{p^*_- - \tau}}}}{\|u\|^{p^- - \tau}} \end{aligned}$$

Therefore, since $\tau < p^-$ we obtain

$$\limsup_{\|u\| \to +\infty} \frac{J(u)}{\Phi(u)} \leq p^+ C_1^\tau \left(M \|\alpha\|_{L^{(\frac{p^*_-}{\tau})'}} + \frac{p^+ A^{\gamma+1-\tau}}{\gamma+1} \|m\|_{L^{\frac{p^*_-}{p^*_- - \tau}}} \right) \varepsilon \tag{3.9}$$

for $\|u\| > 1$ we get

$$\limsup_{\|u\| \to +\infty} \frac{J(u)}{\Phi(u)} \leq p^+ C_1^\tau \left(M \|\alpha\|_{L^{(\frac{p^*_-}{\tau})'}} + \frac{p^+ A^{\gamma+1-\tau}}{\gamma+1} \|m\|_{L^{\frac{p^*_-}{p^*_- - \tau}}} \right) \varepsilon \tag{3.10}$$

Finally, for an arbitrary ε and in view of (3.10) and (3.8), we get

$$\max \left\{ \limsup_{\|x\| \to +\infty} \frac{J(x)}{\Phi(x)}, \limsup_{\|x\| \to 0} \frac{J(x)}{\Phi(x)} \right\} \leq 0$$

Hence, all the assumptions of Theorem 1.1 are satisfied (with $x_0 = 0$).

Moreover, since the function $G : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable in $\mathbb{R}^N$ and C^1 in $\mathbb{R}$ such that $G_u = g$ satisfying (1.4). Then using standard arguments the functional $\Psi(u) = \int_{\mathbb{R}^n} G(x, u)\, dx$ is well defined and continuously Gâteaux differentiable on W, with compact derivative, and one has

$$\Psi'(u)v = \int_{\mathbb{R}^n} g(x, u) v\, dx$$

for all $u, v \in W$. So, by Theorem 1.1, the problem (1.1) has at least two nonzero weak solutions which are critical points of the functional $\Phi - \lambda J - \mu \Psi$.
Then the proof of Theorem 1.3 is achieved.

Example

In this example we consider the problem (1.1), where $N = 5$, $p(x) = cos(|x|) + 3,001$ the one equation problem become,

$$\begin{gathered} -\operatorname{div}(a(x)|\nabla u|^{cos(|x|)+1,001} \nabla u) + b(x)|u|^{cos(|x|)+1,001} u = \lambda f(x, u) + \mu g(x, u) \quad \text{in } \mathbb{R}^5 \\ u \in W_0^{1, cos(|x|)+3,001}(\mathbb{R}^5), \end{gathered} \tag{3.11}$$

$$p^- = 2.001; p^+ = 4.001; p^*_- = 10.01 > 10,$$

The function $p(x)$ satisfy the theorem conditions then for any fonction $m \in L^{\frac{3,001}{2,001}}(\mathbb{R}^5) \cap L^{\frac{\nu}{\nu-1}(\frac{3,001}{4,001-\gamma+1})}(\mathbb{R}^5)$, γ and ν satisfying $4,001 < \gamma + 1 < \nu < 10 < p^*_-$. For the function $F(x, t)$, let choose τ such that $\frac{\gamma+1}{2} < \tau < p^-$, and let $0 < \varepsilon < \min(1, \tau - \frac{\gamma+1}{2})$ be small enough. Consider the function $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$F(x, u) = \frac{1}{\tau - \varepsilon} m(x)|u|^{\tau - \varepsilon} \ln |u| \ \ if u \neq 0, \ \ F(x, 0) = 0$$

Then, condition (1.3) is satisfied since $\ln |u| + \frac{1}{\tau - \varepsilon} < |u|^{\gamma - \tau + 1 + \varepsilon}$. It is not difficult to see that the remaining hypotheses of Theorem 1.2 are satisfied

4 (p(x),q(x))-Laplacian System

In this section we consider the two-equations nonlinear problems involving the $(p(x), q(x))$-laplacian :

$$\begin{aligned} -\operatorname{div}\left(a(x)\nabla u|^{p(x)-2}\nabla u\right) + b(x)|u|^{p(x)-2}u &= \lambda f_u(x,u,v) + \mu g_u(x,u,v) \quad \text{in } \mathbb{R}^N \\ -\operatorname{div}\left(a'(x)|\nabla v|^{q(x)-2}\nabla v\right) + b'(x)|u|^{q(x)-2}u &= \lambda f_v(x,u,v) + \mu g_v(x,u,v) \quad \text{in } \mathbb{R}^N \\ u \in W_0^{1,p(x)}(\mathbb{R}^N), v &\in W_0^{1,q(x)}(\mathbb{R}^N) \end{aligned} \tag{4.1}$$

where $2 < p^- < p(x) < p^+ < N, 2 < q^- < q(x) < q^+ < N$. $\lambda, \mu > 0$ are parameters. The functions $a(x), a'(x)$ are measurable functions in $L^\infty(\mathbb{R}^n)$ with $\inf_{(\mathbb{R}^n)} a'(x)$, $\inf_{(\mathbb{R}^n)} a(x) > 0$. The functions $b(x), b'(x)$ are measurable functions in $L^\infty_{loc}(\mathbb{R}^n)$ with $\inf_{(\mathbb{R}^n)} b'(x)$, $\inf_{(\mathbb{R}^n)} b(x) > 0$. The functions $f(x,t,t')$ and $g(x,t,t')$ having sub-critical growth with respect to t, t' More precisely, we assume that f is a Carathéodory function satisfying the following condition

$$|f(x,t,t')| \leq m(x)\left(|t|^\gamma + |t'|^\gamma\right) \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad \forall t \in \mathbb{R}, \tag{4.2}$$

where $m(x)$ is a positive function such that

$$m \in L^{\frac{s^-}{s^- -1}}(\mathbb{R}^n) \cap L^{\frac{\nu}{\nu-1}\left(\frac{s^-}{s^- -(\gamma+1)}\right)}(\mathbb{R}^n),$$

with

$$s^+ < \gamma + 1 < \nu < s^*_-$$

where

$$s^+ = max(p^+, q^+), \;\; s^- = min(p^-, q^-), \;\; s^*_- = min(p^*_-, q^*_-)$$

with

$$p^*_- = \frac{np^-}{n - p^+}, q^*_- = \frac{nq^-}{n - q^+}.$$

$$\sup_{(x,t,t') \in \mathbb{R}^n \times (\mathbb{R} - \{0\})^2} \frac{|g(x,,t,t')|}{h(x)(|t|^{p^*_-} + |t'|^{p^*_-})} < +\infty, \tag{4.3}$$

for some

$$h \in L^\infty(\mathbb{R}^n) \;\; and \;\; g(x,0) = 0.$$

We shall look for a weak-solution of (4.1) in the space $W_0^{1,p(x)}(\mathbb{R}^N) \times W_0^{1,q(x)}(\mathbb{R}^N)$ which is endowed with the Cartesian norm $\|(u,v)\| = \|u\|_{W_0^{1,p(x)}(\mathbb{R}^N)} + \|v\|_{W_0^{1,q(x)}(\mathbb{R}^N)}$. We set

$E = W_0^{1,p(x)}(\mathbb{R}^N)$, $E' = W_0^{1,q(x)}(\mathbb{R}^N)$, $U = (u,v)$, $f(x,U) = f_u(x,u,v) + f_v(x,u,v)$, $g(x,U) = g_u(x,u,v) + g_v(x,u,v)$, and $F(x,u,v) = \int_0^u f_u(x,s,v)\,ds + \int_0^v f_v(x,u,s)\,ds$

Theorem 4.1. *Let us suppose for $1 < \tau < s^-$ and some positive function $\alpha(x) \in L^{(\frac{s^*_-}{\tau})'}(\mathbb{R}^n)$ that*

$$\limsup_{|U| \to +\infty} \frac{F(x,U)}{\alpha(x)|U|^\tau} \leq M < +\infty \quad uniformly \quad x \in \mathbb{R}^N, \tag{4.4}$$

and

$$\sup_{U \in E \times E'} \left(\int_{\mathbb{R}^N} F(x,U)dx\right) > 0$$

If we set

$$\omega = \frac{1}{s^+} \inf\left\{\frac{\int_{\mathbb{R}^n} A(x,U)dx}{\int_{\mathbb{R}^n} F(x,U)\,dx} : U \in W, \right\}, \tag{4.5}$$

where $A(x,U) = \Big(a(x)|\nabla u|^{p(x)} + a'(x)|\nabla u|^{q(x)} + b(x)|u|^{p(x)}dx + b'(x)|u|^{q(x)}\Big)$ *and* $W = \{U \in E \times E', \int_{\mathbb{R}^n} F(x,U)\,dx > 0\}$*, then for each compact interval* $[a,b] \subset]\omega, +\infty[$*, there exists* $r_1 > 0$ *with the following property: for every* $\lambda \in [a,b]$*, and every function* $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *which is measurable in* $\mathbb{R}^N$ *and continuous in* $\mathbb{R}$ *satisfying* (4.3) *, there exists* $\delta > 0$ *such that for each* $\mu \in [0,\delta]$*, the two-equation system* (4.1) *has at least two nonzero weak solutions in* E *whose norms are less than* r_1*.*

To prove this result we need to define the following functions
$\Phi, J, \Psi : E \times E' \longrightarrow \mathbb{R}$

$$\Phi(U) = \int_{\mathbb{R}^n} \frac{1}{p(x)} \Big(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\Big)\,dx + \int_{\mathbb{R}^n} \frac{1}{q(x)} \Big(|\nabla v|^{q(x)} + b'(x)|v|^{q(x)}\Big)\,dx$$

and

$$J(U) = \int_{\mathbb{R}^n} F(x,U)\,dx$$

and

$$\Psi(U) = \int_{\mathbb{R}^n} G(x,U)\,dx$$

The functions $\Phi(U)$, $J(U)$, $\Psi(U)$ need to satisfy the conditions of the theorem 1.1, i.e

$\Phi : E \times E' \to \mathbb{R}$, a coercive, sequentially weakly lower semi continuous C^1 functional, belonging to $\mathcal{W}_{E\times E'}$, bounded on each bounded subset of $E \times E'$, and whose derivative admits a continuous inverse on X^*; also it not difficult to see that $\phi(U)$ satisfy;

$$\frac{\|u\|_E^{p(x)}}{p^+} + \frac{\|v\|_{E'}^{q(x)}}{q^+} \le \phi(U) \le \frac{\|u\|_E^{p(x)}}{p^-} + \frac{\|v\|_{E'}^{q(x)}}{q^-}$$

then for $\|u\|_E \le 1$ *and* $\|v\|_{E'} \le 1$ we have:

$$\frac{\|u\|_E^{p^+} + \|v\|_{E'}^{q^+}}{s^+} \le \Phi(U) \le \frac{\|u\|_E^{p^-} + \|v\|_{E'}^{q^-}}{s^-}$$

then for a very small value of $\|u\|_E$, *and*, $\|v\|_{E'}$ we have

$$\frac{(\|u\|_E + \|v\|_{E'})^{s^+}}{s^+} \le \Phi(U) \le \frac{(\|u\|_E + \|v\|_{E'})^{s^-}}{s^-}$$

witch leads us to an inequality similar to the one (3.3) in section 3,

$$\frac{(\|U\|_{E\times E'})^{s^+}}{s^+} \le \Phi(U) \le \frac{(\|U\|_{E\times E'})^{s^-}}{s^-} \tag{4.6}$$

Olso for a very big value of $\|u\|_E$ and $\|v\|_{E'}$, we have an inequality similar (3.4) in section 3

$$\frac{(\|U\|_{E\times E'})^{s^-}}{s^+} \le \Phi(U) \le \frac{(\|U\|_{E\times E'})^{s^+}}{s^-} \tag{4.7}$$

Those inequalities (4.6), (4.7) will play a major role (similar to the one played by (3.3) and (3.4) in section 3) to prove

$$\alpha = \max\Big\{0, \limsup_{\|U\|\to+\infty} \frac{J(U)}{\Phi(U)}, \limsup_{\|U\|\to 0} \frac{J(U)}{\Phi(U)}\Big\},$$

$$\beta = \sup_{x\in\Phi^{-1}(]0,+\infty[)} \frac{J(U)}{\Phi(U)},$$

with $\alpha = 0 < \beta = \infty$ Then, for each compact interval $[a,b] \subset]0,\infty[$ there exists $r_1 > 0$ such that the problem (4.1) has at least three solutions in $E \times E'$ whose norms are less than r_1.

Competing Interests

The authors declare that no competing interests exist.

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