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Some Applications of Bipolar Soft Set: Characterizations of Two Isomorphic Hemi-Rings via *BSI*-*h*-Ideals

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Abstract

In this study, we intended to present insights into bipolar soft set in the union of two isomorphic hemi-rings. This concept provides a new soft algebraic tool in many uncertainties problems. We introduced BS-h-sum and BS-h-product of BS-sets. In particular, we discussed bipolar soft intersectional h-ideals in the union of two isomorphic hemi-rings. In addition, we characterized isomorphic h-hemi-regular hemi-rings using bipolar soft intersectional h-ideals.

Keywords: Hemi-rings, BS-h-sum, BS-h-product, bipolar soft intersectional hemi-rings, bipolar soft intersectional h-ideals, h-hemi-regular hemi-rings.

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1 Introduction

There are various problems of information sciences, industrial sciences, engineering, medical sciences and social sciences etc. containing vagueness and uncertainty. Molodtsov [1], popularized the soft sets as a new soft tool in mathematics for dealing with uncertainties. Subsequently, this theory has

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been applied in many research areas such as data analysis, approximate reasoning, simulation and decision-making etc. (see [2]-[6]). Maji et al. [7], proposed some useful definitions in soft set theory. Nowadays, research in this area advancing rapidly with remarkable applications. Cagman et al. (see [8]- [10]), have been applied this theory in various algebraic structure, especially in the theory of groups. Ali et al. [11],[12], advanced soft set with useful new operations. The bipolar soft set with applications in decision-making popularized by Shabir et al. [13], and discussed exhaustively by Karaaslan et al. [14].

Semi-rings presented by Vandiver [15] and theory of semi-rings have been researched by many researchers [16]-[18]. Ideals of hemi-rings, as an especial class of particular hemi-rings, play an essential role in the algebraic structure theories any how many properties of hemi-rings are described by ideals. Henriksen [19], defined ideals in hemi-rings. In 2013, fuzzy ideals and interior ideals in ordered semi-rings investigated by Mandal [20]. Characterizations of hemi-rings by their bipolar-valued fuzzy h-ideals introduced by Mahmood et al. [21]. Feng et al. [22], initiated soft semi-rings. Recently, Zhan et al. [23]-[25], investigated soft union set and soft intersectional set which give new soft tool to consider problems that containing vagueness. Moreover, they characterized hemi-rings, h-semi-simple hemi-rings and h-quasi-hemi-regular hemi-rings by using soft union h-ideals and soft intersectional h-ideals. Also (see [26]- [28]).

A bipolar soft set is obtained by considering not only a carefully chosen set of parameters but also an allied set of oppositely meaning parameters. Structure of a bipolar soft set is managed by two functions, say $\widetilde{\gamma} : R \longrightarrow P(U)$ and $\widetilde{\delta} : S \longrightarrow P(U)$, we define an isomorphic mapping h from R to S. For example, we consider two sets mania patient set $R = \{c_1, c_4, c_8\}$ and depression patient set $S = \{c_2, c_3, c_5\}$ with mapping $h(c_1) = c_3, h(c_4) = c_5$ and $h(c_8) = c_5$,then we have bipolar soft sets $\langle\widetilde{\gamma}(c_1), \widetilde{\delta}(c_3)\rangle, \langle\widetilde{\gamma}(c_4), \widetilde{\delta}(c_5)\rangle$ and $\langle\widetilde{\gamma}(c_1), \widetilde{\delta}(c_3)\rangle$ over U such that $\widetilde{\gamma}(c_1) \cap \widetilde{\delta}(c_3) = \emptyset$ and so on. The function $\widetilde{\delta}$ describes somewhat an opposite or negative approximation for the attractiveness of a things relative to the approximation computed by $\widetilde{\gamma}$. Maji et al. [7] had used the "not set" to define the complement of a soft set. Above representation $\widetilde{\delta}$ is more generalized then soft complement function. In [13] authors used a bijective mapping from any set R to S, but in algebraic structures isomorphism rather more generalized.

In this study, we intended to present insights into bipolar soft set in the union of two isomorphic hemi-rings. This notion should evolve a foundation to handle problems in algebraic structures. In the manner of Ma and Zhan [25], we investigated bipolar soft intersectional h-ideals (breifly BSI-h-ideals) in the union of two isomorphic hemi-rings and some useful results. We introduced BS-h-sum and BS-h-product of BS-sets. Finally, we characterized isomorphic h-hemi-regular hemi-rings using bipolar soft intersectional h-ideals.

2 Preliminaries

A semi-ring is a system consisting of non-empty set R on which operation of addition and multiplication (denoted in usual manner) have been defined such that $(R, +)$, and $(R, .)$ are semi-group and multiplication distributes over addition either side. An element $0 \in R$, such that $0.x = x.0 = 0$ and $0 + x = x + 0 = x$ for all $x \in R$ is called zero element. An ideal of a semi-ring R is a subset L of R which is closed under addition and $RL \subseteq L, LR \subseteq L$. A semi-ring $(R, +, .)$ with zero is called a hemi-ring if $(R, +)$ is commutative.

Let R and S be two hemi-rings then a mapping η from a hemi-ring R to a hemi-ring S is said to be an isomorphism of hemi-rings if it holds, *(i)* $\eta(0_R) = 0_S$ for all $0_R \in R, 0_S \in S$ *(ii)* $\eta(x_R + y_R) = \eta(x_R) + \eta(y_R)$ and $\eta(x_R y_R) = \eta(x_R)\eta(y_R)$ for all $x_R, y_R \in R$ *(iii)* η is bijective. Moreover, for basic definitions and notions of hemi-rings we refer to [18].

Let U be an universe, and E be a parameters set. Let $P(U)$ be a set of power set of U and L, L_1, L_2 be non-empty subsets of E. Then we define 2.1 and 2.2 as follow.

2.1 Definition[7]

A soft set $\widetilde{G}_L$ over U is defined as $\widetilde{G}_L : E \to P(U)$ such that $\widetilde{G}_L(a) = \emptyset$ if $a \notin L$. It can be represented by $\widetilde{G}_L = \{(a, \widetilde{G}_L(a)) \mid a \in E, \widetilde{G}_L(a) \in P(U)\}$.

Moreover, $\widetilde{G}_L$ is also called an approximate function.

The set of all soft set over U will be denoted by $\widetilde{S(U)}$.

2.2 Definition[10] , [29] , [30]

Let $\widetilde{G}_{L_1}, \widetilde{G}_{L_2}, \widetilde{G}_L \in \widetilde{S(U)}$

(i) A soft set is called *null soft set* if $\widetilde{G}_L(a) = \emptyset$, for $a \in E$. It denoted by $\widetilde{\Phi}_L$.

(ii) A soft set is called *whole soft set* if $\widetilde{G}_L(a) = U$, for $a \in E$. It denoted by $\widetilde{\mu}_L$.

(iii) Soft set $\widetilde{G}_{L_1}$ is called *subset* of $\widetilde{G}_{L_2}$, denoted by $\widetilde{G}_{L_1} \widetilde{\subseteq} \widetilde{G}_{L_2}$ and defined by $\widetilde{G}_{L_1}(a) \subseteq \widetilde{G}_{L_2}(a)$ for all $a \in E$.

(iv) *Union* of $\widetilde{G}_{L_1}$ and $\widetilde{G}_{L_2}$ denoted by $\widetilde{G}_{L_1 \widetilde{\cup} L_2} = \widetilde{G}_{L_1} \widetilde{\cup} \widetilde{G}_{L_2}$ and defined by $\widetilde{G}_{L_1 \widetilde{\cup} L_2}(a) = \widetilde{G}_{L_1}(a) \cup \widetilde{G}_{L_2}(a)$ for all $a \in E$.

(v) *Intersection* of $\widetilde{G}_{L_1}$ and $\widetilde{G}_{L_2}$ denoted by $\widetilde{G}_{L_1 \widetilde{\cap} L_2} = \widetilde{G}_{L_1} \widetilde{\cap} \widetilde{G}_{L_1}$ and defined by $\widetilde{G}_{L_1 \widetilde{\cap} L_2}(a) = \widetilde{G}_{L_1}(a) \cap \widetilde{G}_{L_2}(a)$ for all $a \in E$.

(vi) *Complement* of $\widetilde{G}_L$ denoted $(\widetilde{G}_L)^c$ or $\widetilde{G}_{\neg L}$ and defined by $\widetilde{G}_{\neg L}(a) = P(U) - \widetilde{G}_L(a)$ for all $a \in E$.

(vii) *Upper* $\widetilde{\alpha}$*-inclusion* of $\widetilde{G}_L$ is denoted by $(\widetilde{G}^u_L : \widetilde{\alpha})$ and defined by $(\widetilde{G}^u_L : \widetilde{\alpha}) = \{x \in A \mid \widetilde{G}_L(x) \supseteq \widetilde{\alpha}\}$ for all $x \in E, \widetilde{\alpha} \subseteq U$.

2.3 Definition[25]

A soft set $\widetilde{G}_R$ over U is called a soft intersection h-ideals in hemi-rings R over U if it holds

(si_1) $\widetilde{G}_R(a + b) \supseteq \widetilde{G}_R(a) \cap \widetilde{G}_R(b)$,

(si_2) $\widetilde{G}_R(ab) \supseteq \widetilde{G}_R(a)$,

(si_3) $\widetilde{G}_R(ab) \supseteq \widetilde{G}_R(b)$,

(si_4) $\widetilde{G}_R(a) \supseteq \widetilde{G}_R(x_1) \cap \widetilde{G}_R(x_2)$, with $a + x_1 + c = x_2 + c$, for $a, c, x_1, x_2 \in R$.

2.4 Definition[14]

Let E be the parameters set with $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. There is a bijective mapping $h : E_1 \to E_2$. If $\widetilde{F} : E_1 \to P(U)$ and $\widetilde{G} : E_2 \to P(U)$ are two mappings such that $\widetilde{F}(e) \cap \widetilde{G}(h(e)) = \emptyset$, then the triple $(\widetilde{F}, \widetilde{G}, E)$ is called bipolar soft set (i.e. briefly BS-set) over U. For the sake of simplicity, we shall use the symbol $\widetilde{B} = (\widetilde{F}, \widetilde{G})$. We represents it as following form

$$(\widetilde{F}, \widetilde{G}) = \left\{ \begin{array}{c} \langle (x, \widetilde{F}_{E_1}(x)), (h(x), \widetilde{G}_{E_2}(h(x))) \rangle : x \in R \\ \text{such that } \widetilde{F}_{E_1}(x) \cap \widetilde{G}_{E_2}(h(x)) = \emptyset \end{array} \right\}$$

If $\widetilde{F}_{E_1}(x)) = \emptyset$ and $\widetilde{G}_{E_2}(h(x)) = \emptyset$ for $x \in R$, then $\langle (x, \emptyset), (h(x), \emptyset)) \rangle$ is nothing remarked in BS-set.

3 Main Results

In this section, we presented bipolar soft set in union of two isomorphic hemi-rings. We contemplated *BS-h-product*, *BS-h-sum and BS*-characteristic function.

3.1 Definition

Let E be the parameters set and R and S be two hemi-rings such that $E = R \cup S$ and $R \cap S \neq \emptyset$ and mapping $\eta : R \to S$ be an isomorphism. If $\widetilde{\gamma}_R : R \to P(U)$ and $\widetilde{\delta}_S : S \to P(U)$ are two mapings such that $\widetilde{\gamma}_R(e) \cap \widetilde{\delta}_S(\eta(e)) = \emptyset$, then the triple $(\widetilde{\gamma}_R, \widetilde{\delta}_S, E)$ is called bipolar soft set (BS-set) in union of two hemi-rings over U. For the sake of simplicity, we shall use the symbol $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ and represents it as following form:

$$(\widetilde{\gamma}_R, \widetilde{\delta}_S) = \left\{ \begin{array}{c} \langle (x, \widetilde{\gamma}_R(x)), (\eta(x), \widetilde{\delta}_S(\eta(x))) \rangle : x \in R \\ \text{such that } \widetilde{\gamma}_R(x) \cap \widetilde{\delta}_S(\eta(x)) = \emptyset \end{array} \right\}.$$

If $\widetilde{\gamma}_R(x)) = \emptyset$ and $\widetilde{\delta}_S(\eta(x)) = \emptyset$ for $x \in R$, then $\langle (x, \emptyset), (\eta(x), \emptyset)) \rangle$ is nothing written in bipolar soft set. We shall denote the set of all bipolar soft sets of $R \cup S$ over U by $BS(R \cup S)$.

Moreover, throughout this paper we will denote set of parameters $E = R \cup S$ as an union of two hemi-rings R and S with $R \cap S = \emptyset$ and mapping η will be an isomorphism between R and S. Otherwise, we particularized.

3.2 Definition

Let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$,$\widetilde{B}^* = (\widetilde{\gamma}^*_R, \widetilde{\delta}^*_S) \in BS(R \cup S)$ then

(i) BS-set $\widetilde{B}$ is *subset* of BS-set $\widetilde{B}^*$ denoted by $\widetilde{B} \sqsubseteq \widetilde{B}^*$ or $\widetilde{\gamma}_R \sqsubseteq \widetilde{\gamma}^*_R, \widetilde{\delta}_S \sqsubseteq \widetilde{\delta}^*_S$ if $(\widetilde{\gamma}_R(x) \subseteq \widetilde{\gamma}^*_R(x), (\widetilde{\delta}_S(\eta(x)) \supseteq \widetilde{\delta}^*_S(\eta(x))$ for all $x \in R, \eta(x) \in S$.

(ii) BS-set $\widetilde{B}$ is *equal* to BS-set $\widetilde{B}^*$ denoted by $\widetilde{B} = \widetilde{B}^*$ if $\widetilde{B} \sqsubseteq \widetilde{B}^*, \widetilde{B} \sqsupseteq \widetilde{B}^*$.

(iii) *Intersection* of $\widetilde{B}$ and $\widetilde{B}^*$ is denoted by $\widetilde{B} \sqcap \widetilde{B}^* = (\widetilde{\gamma}_R \sqcap \widetilde{\gamma}^*_R, \widetilde{\delta}_S \sqcap \widetilde{\delta}^*_S)$ and defined by $\widetilde{B} \sqcap \widetilde{B}^*(x) = \Big((\widetilde{\gamma}_R \sqcap \widetilde{\gamma}^*_R)(x), (\widetilde{\delta}_S \sqcap \widetilde{\delta}^*_S)(\eta(x)) \Big)$ where $(\widetilde{\gamma}_R \sqcap \widetilde{\gamma}^*_R)(x) = (\widetilde{\gamma}_R(x) \cap \widetilde{\gamma}^*_R(x))$ and $(\widetilde{\delta}_S \sqcap \widetilde{\delta}^*_S)(\eta(x)) = (\widetilde{\delta}_S(\eta(x)) \cup \widetilde{\delta}^*_S(\eta(x)))$ for all $x \in R, \eta(x) \in S$.

(iv) *Union* of $\widetilde{B}$ and $\widetilde{B}^*$ is denoted by $\widetilde{B} \sqcup \widetilde{B}^* = (\widetilde{\gamma}_R \sqcup \widetilde{\gamma}^*_R, \widetilde{\delta}_S \sqcup \widetilde{\delta}^*_S)$ and defined by $\widetilde{B} \sqcup \widetilde{B}^*(x) = \Big((\widetilde{\gamma}_R \sqcup \widetilde{\gamma}^*_R)(x), (\widetilde{\delta}_S \sqcup \widetilde{\delta}^*_S)(\eta(x)) \Big)$ where $(\widetilde{\gamma}_R \sqcup \widetilde{\gamma}^*_R)(x) = (\widetilde{\gamma}_R(x) \cup \widetilde{\gamma}^*_R(x))$ and $(\widetilde{\delta}_S \sqcup \widetilde{\delta}^*_S)(\eta(x)) = (\widetilde{\delta}_S(\eta(x)) \cap \widetilde{\delta}^*_S(\eta(x)))$ for all $x \in R, \eta(x) \in S$.

(v) $\widetilde{B}$ is called *null* BS-set if $\widetilde{\gamma}_R(x) = \widetilde{\emptyset}_R$, $\widetilde{\delta}_S(\eta(x)) = \widetilde{U}_S$ denoted by $\widetilde{\Phi} = (\widetilde{\emptyset}_R, \widetilde{U}_S)$.

(vi) $\widetilde{B}$ is called *absolute* BS-set if $\widetilde{\gamma}_R(x) = \widetilde{U}_S$, $\widetilde{\delta}_S(\eta(x)) = \widetilde{\emptyset}_S$ denoted by $\widetilde{U} = (\widetilde{U}_R, \widetilde{\emptyset_S})$.

3.3 Example

Let $U = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be Universe and $E = R \cup S$ where $R = \{x_1, x_2, x_3\}$ and $S = \{y_1, y_2, y_3\}$ with be two hemi-rings. Define $\eta : R \to S$ with $\eta(x_1) = y_2, \eta(x_2) = y_3, \eta(x_3) = y_1$

So, we can present two BS-set in $R \cup S$ over U by

$$(\widetilde{\gamma}_R, \widetilde{\delta}_S) = \left\{ \begin{array}{c} \langle (x_1, \{v_1, v_2\}), (\eta(x_1), \{v_5, v_6, v_7\}) \rangle \\ \langle (x_2, \{v_1, v_3, v_5\}), (\eta(x_2), \{v_2, v_4, v_7\}) \rangle \\ \langle (x_1, \{v_2, v_4\}), (\eta(x_2), \{v_3, v_6\}) \rangle \end{array} \right\},$$

$$(\widetilde{\gamma}_R^*, \widetilde{\delta}_S^*) = \left\{ \begin{array}{c} \langle (x_1, \{v_1, v_2, v_7\}), (\eta(x_2), \{v_4, v_6\}) \rangle \\ \langle (x_2, \{v_2, v_3, v_5\}), (\eta(x_2), \{v_1, v_4, v_7\}) \rangle \\ \langle (x_1, \{v_1, v_4\}), (\eta(x_2), \{v_3, v_7\}) \rangle \end{array} \right\}$$

3.4 Definition

Let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ and $\widetilde{B}^* = (\widetilde{\gamma}_R^*, \widetilde{\delta}_S^*)$ be two BS-set in $R \cup S$ over U. We define

(i) **BS-h-sum**

$(\widetilde{B} \oplus_h \widetilde{B}^*) = ((\widetilde{\gamma}_R \oplus_h \widetilde{\gamma}_R^*), (\widetilde{\delta}_S \oplus_h \widetilde{\delta}_S^*))$

where

$$(\widetilde{\gamma}_R \oplus_h \widetilde{\gamma}_R^*)(a) = \bigcup_{a+x_1+y_1+b=x_2+y_2+b} \{\widetilde{\gamma}_R(x_1) \cap \widetilde{\gamma}_R^*(x_2) \cap \widetilde{\gamma}_R(y_1) \cap \widetilde{\gamma}_R^*(y_2)\}$$

$$(\widetilde{\delta}_S \oplus_h \widetilde{\delta}_S^*)(\eta(a)) = \bigcap_{\substack{\eta(a)+\eta(x_1)+\eta(y_1)+\eta(b) \\ =\eta(x_2)+\eta(y_2)+\eta(b)}} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(x_1)) \cup \widetilde{\delta}_S^*(\eta(x_2)) \\ \cup \widetilde{\delta}_S(\eta(y_1)) \cup \widetilde{\delta}_S^*(\eta(y_2)) \end{array} \right\}$$

$(\widetilde{B} \diamond_h \widetilde{B}^*)(a) = (\widetilde{\emptyset}_R, \widetilde{U}_S)$ if a cannot signified as $a + \sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b$, $\eta(a) + \sum_{k=1}^{m} \eta(y_k)\eta(z_k) + \eta(b) = \sum_{k'=1}^{n} \eta(y'_{k'})\eta(z'_{k'}) + \eta(b)$ $\forall a, y_k, z_{k'}, y'_{k'}, z'_{k'}, b \in R$ and $\eta(a), \eta(y_k), \eta(z_k), \eta(b), \eta(y'_{k'}), \eta(z'_{k'}) \in S$.

(ii) **BS-h-product**

$(\widetilde{B} \diamond_h \widetilde{B}^*) = ((\widetilde{\gamma}_R \diamond_h \widetilde{\gamma}_R^*), (\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S^*))$

where

$$(\widetilde{\gamma}_R \diamond_h \widetilde{\gamma}_R^*)(a) = \bigcup_{a+\sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b} \{\widetilde{\gamma}_R(y_k) \cap \widetilde{\gamma}_R^*(z_k) \cap \widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}_R^*(z'_{k'})\}$$

$$(\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S^*)(\eta(a)) = \bigcap_{\substack{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(b) \\ =\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(b)}} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(y_k)) \cup \widetilde{\delta}_S^*(\eta(z_k)) \\ \cup \widetilde{\delta}_S(\eta(y'_{k'})) \cup \widetilde{\delta}_S^*(\eta(z'_{k'})) \end{array} \right\}$$

$(\widetilde{B} \diamond_h \widetilde{B}^*)(a) = (\widetilde{\emptyset}_R, \widetilde{U}_S)$ if a cannot signified as $a+x_1+y_1+b = x_2+y_2+b$, $\eta(a)+\eta(x_1)+\eta(y_1)+\eta(b) = \eta(x_2) + \eta(y_2) + \eta(b) \forall a, x_1, x_2, y_1, y_2, b \in R$ and $\eta(a), \eta(x_1), \eta(x_2), \eta(b), \eta(y_1), \eta(y_2) \in S$.

3.5 Definition

Let $\emptyset \neq L_1$ and $\emptyset \neq L_2$ be two subset of E such that $L_1 \subseteq R$ and $L_2 \subseteq S$. We denote by $\widetilde{\chi}_{L_1 \widetilde{\longrightarrow} L_2} = (\widetilde{\omega}_{L_1}, \widetilde{\varkappa}_{L_2})$ the BS-characteristic function of $L_1 \widetilde{\longrightarrow} L_2$ over U and define as

$$\widetilde{\omega}_{L_1}(x) = \left\{ \begin{array}{lll} \emptyset_R & \text{if} & x \notin L_1, \\ U_R & \text{if} & x \in L_1 \end{array} \right.$$

$$\widetilde{\varkappa}_{L_2}(\eta(x)) = \left\{ \begin{array}{lll} \emptyset_S & \text{if} & \eta(x) \in L_2, \\ U_S & \text{if} & \eta(x) \notin L_2 \end{array} \right.$$

3.6 Proposition

Let L,$M \subseteq R$, L^*,$M^* \subseteq S$ and mapping $\eta : R \to S$ be an isomorphism. Also L is isomorphic to L^* and M is isomorphic to M^*. Then the following hold:

(1) $L \subseteq M$ and $L^* \subseteq M^* \Longrightarrow \widetilde{\omega}_L \sqsubseteq \widetilde{\omega}_M$ and $\widetilde{\varkappa}_{L^*} \sqsubseteq \widetilde{\varkappa}_{M^*}$(i.e. $\widetilde{\chi}_{L \widetilde{\longrightarrow} L^*} \sqsubseteq \widetilde{\chi}_{M \widetilde{\longrightarrow} M^*}$)

(2) $\widetilde{\chi}_{L \rightrightarrows L^*} \sqcap \widetilde{\chi}_{M \rightrightarrows M^*} = (\widetilde{\omega}_L \sqcap \widetilde{\omega}_M, \widetilde{\varkappa}_{L^*} \sqcap \widetilde{\varkappa}_{M^*})$ such that $\widetilde{\omega}_L \sqcap \widetilde{\omega}_M = \widetilde{\omega}_{L \cap M}$ and $\widetilde{\varkappa}_{L^*} \sqcap \widetilde{\varkappa}_{M^*} = \widetilde{\varkappa}_{L^* \cup M^*}$

(3) $\widetilde{\chi}_{L \rightrightarrows L^*} \diamond_h \widetilde{\chi}_{M \rightrightarrows M^*} = (\widetilde{\omega}_{\overline{LM}}, \widetilde{\varkappa}_{\overline{L^* M^*}})$.

4 BSI-hemi-rings

In this section, we initiated bipolar soft intersectional hemi-rings and presented some related properties.

4.1 Definition

A BS-set $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ over U is called a bipolar soft intersectional hemi-ring (i.e, briefly BSI-hemi-ring) in union of two isomorphic hemi-rings R and S over U if it holds

(B-Sh_1) $\widetilde{\gamma}_R(a+b) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b), \widetilde{\delta}_S(\eta(a+b)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b))$

(B-Sh_2) $\widetilde{\gamma}_R(ab) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b), \widetilde{\delta}_S(\eta(ab)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b))$

(B-Sh_3) $\widetilde{\gamma}_R(a) \supseteq \widetilde{\gamma}_R(x_1) \cap \widetilde{\gamma}_R(x_2), \widetilde{\delta}_S(\eta(a)) \subseteq \widetilde{\delta}_S(\eta(x_1)) \cup \widetilde{\delta}_S(\eta(x_2))$ with $a + x_1 + c = x_2 + c, \eta(a) + \eta(x_1) + \eta(c) = \eta(x_2) + \eta(c)$ for $a, c, x_1, x_2 \in R$, and $\eta(a), \eta(c), \eta(x_1), \eta(x_1) \in S$.

In rest of the paper, set of all BSI-h-hemi-ring of $R \cup S$ over U is denoted by $BSIH(R \cup S)$.

4.2 Example

Let $\mathbb{Z}_6 = R = \{0_R, 1_R, 2_R, 3_R, 4_R, 5_R\}$ be a hemi-ring of non-negative integers module 6.

Also let $S = \{0_S, a_S, b_S, c_S, d_S, e_S\}$ be a hemi-ring with "+" and "." as fallow

$+$	0_S	a_S	b_S	c_S	d_S	e_S
0_S	0_S	a_S	b_S	c_S	d_S	e_S
a_S	a_S	b_S	c_S	d_S	e_S	0_S
b_S	b_S	c_S	d_S	e_S	0_S	a_S
c_S	c_S	d_S	e_S	0_S	a_S	b_S
d_S	d_S	e_S	0_S	a_S	b_S	c_S
e_S	e_S	0_S	a_S	b_S	c_S	d_S

.	0_S	a_S	b_S	c_S	d_S	e_S
0_S	0_S	0_S	0_S	0_S	0_S	0_S
a_S	0_S	a_S	b_S	c_S	d_S	e_S
b_S	0_S	b_S	d_S	0_S	b_S	d_S
c_S	0_S	c_S	0_S	e_S	0_S	c_S
d_S	0_S	d_S	b_S	0_S	b_S	b_S
e_S	0_S	e_S	d_S	c_S	b_S	a_S

Here $E = R \cup S$ and clearly $R \cap S = \emptyset$. Assume $U = \mathbb{Z}_6$ and we describe a BS-set $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ in $R \cup S$ over U. There is an isomorphism $\eta : R \to S$ such that $\eta(0_R) = 0_S, \eta(1_R) = a_S, \eta(2_R) = b_S, \eta(3_R) = c_S, \eta(4_R) = d_S, \eta(5_R) = e_S$.

	0_R	1_R	2_R	3_R	4_R	5_R
$\widetilde{\gamma}_R(x)$	$\{0, 2, 3, 4, 5\}$	$\{0, 4, 5\}$	$\{4, 5\}$	$\{2, 4\}$	$\{4, 5\}$	$\{0, 4, 5\}$
$\widetilde{\delta}_R(\eta(x))$	$\{1\}$	$\{1, 2, 3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{1, 2\}$	$\{1, 2, 3\}$

Clearly $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIH(R \cup S)$.

4.3 Proposition

If $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIH(R \cup S)$, then $\widetilde{\gamma}_R(0_R) \supseteq \widetilde{\gamma}_R(x_R)$ and $\widetilde{\delta}_S(\eta(0_R)) \subseteq \widetilde{\delta}_S(\eta(x_R))$ $\forall 0_R, x_R \in R, \eta(0_R), \eta(x_R) \in S$.

4.4 Proposition

Two non-empty subsets L of R and M of S are h-subhemi-rings of R and S respectively if and only if BS-set $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ of $R \cup S$ over U is defined by

$$\widetilde{\gamma}_R(x) = \begin{cases} \alpha & \text{if } x \in R/L \\ \alpha' & \text{if } x \in L \end{cases}$$

$$\widetilde{\delta}_S(\eta(x)) = \begin{cases} \beta & \text{if } \eta(x) \in M \\ \beta' & \text{if } \eta(x) \in S/M \end{cases}$$

is a BSI-hemi-ring of $R \cup S$ over U, with α,$\alpha', \beta, \beta' \subseteq U$ and $\alpha \subseteq \alpha'$, $\beta \subseteq \beta'$.

Proof. Assume that two non-empty subsets L of R and M of S are h-subhemi-rings of R and S respectively. Let $a, b \in R$ and $\eta(a), \eta(b) \in S$.

(i) If $a, b \in L$ and $\eta(a), \eta(b) \in M$, then $ab, a + b \in L$ and $\eta(a)\eta(b), \eta(a) + \eta(b) \in M$. Therefore, $\widetilde{\gamma}_R(a) = \widetilde{\gamma}_R(b) = \widetilde{\gamma}_R(a + b) = \widetilde{\gamma}_R(ab) = \alpha'$ and $\widetilde{\delta}_S(\eta(a)) = \widetilde{\delta}_S(\eta(b)) = \widetilde{\delta}_S(\eta(a + b)) = \widetilde{\delta}_S(\eta(ab)) = \beta$ which implies $\widetilde{\gamma}_R(a + b) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b), \widetilde{\delta}_S(\eta(a + b)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b))$ and $\widetilde{\gamma}_R(ab) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b), \widetilde{\delta}_S(\eta(ab)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b))$

(ii) If either one if a and b does not belong to L, then either one if $\eta(a)$ and $\eta(b)$ does not belong to M. So that $ab \in L$ or $ab \notin L, a + b \in L$ or $a + b \notin L$ and $\eta(a)\eta(b) \in M$ or $\eta(a)\eta(b) \notin M, \eta(a) + \eta(b) \in M$ or $\eta(a) + \eta(b) \notin M$. Therefore,

$\widetilde{\gamma}_R(a + b) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b) = \alpha, \widetilde{\delta}_S(\eta(a + b)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b)) = \beta'$ and $\widetilde{\gamma}_R(ab) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b) = \alpha, \widetilde{\delta}_S(\eta(ab)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b)) = \beta'$

(iii) Moreover, for $a, x, y, b \in R$ we have $a + x + b = y + b$. If $x, y \in L, \eta(x), \eta(y) \in M$ then $a \in L, \eta(a) \in M$ with,

$\widetilde{\gamma}_R(a) = \widetilde{\gamma}_R(x) \cap \widetilde{\gamma}_R(y) = \alpha', \widetilde{\delta}_S(\eta(a)) = \widetilde{\delta}_S(\eta(x)) \cup \widetilde{\delta}_S(\eta(y)) = \beta$. If $a \notin L$ or $b \notin L$, then $\widetilde{\gamma}_R(a) \supseteq \widetilde{\gamma}_R(x) \cap \widetilde{\gamma}_R(y) = \alpha, \widetilde{\delta}_S(\eta(a)) \subseteq \widetilde{\delta}_S(\eta(x)) \cup \widetilde{\delta}_S(\eta(y)) = \beta'$.

Therefore, $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIH(R \cup S)$.

Conversely, let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIH(R \cup S)$ and for $a, b \in R, \eta(a), \eta(b) \in S$ $\widetilde{\gamma}_R(a + b) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b) = \alpha', \widetilde{\delta}_S(\eta(a + b)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b)) = \beta$ and $\widetilde{\gamma}_R(ab) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b) = \alpha', \widetilde{\delta}_S(\eta(ab)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b)) = \beta$ which implies $a + b, ab \in L$ and $\eta(a + b), \eta(ab) \in M$.

Now, for $x, y \in R, \eta(x), \eta(y) \in S$ and $a, b \in L, \eta(a), \eta(b) \in M$ we have $a + x + b = y + b$, $\eta(a) + \eta(x) + \eta(b) = \eta(y) + \eta(b)$ such that

$\widetilde{\gamma}_R(a) = \widetilde{\gamma}_R(x) \cap \widetilde{\gamma}_R(y) = \alpha', \widetilde{\delta}_S(\eta(a)) = \widetilde{\delta}_S(\eta(x)) \cup \widetilde{\delta}_S(\eta(y)) = \beta$ which implies $a \in L, \eta(a) \in M$.

Hence subsets L of R and M of S are h-subhemi-rings of R and S respectively. □

4.5 Corollary

Let $\emptyset \neq L$ and $\emptyset \neq M$ be two subset of E such that $L \subseteq R, M \subseteq S$. Then L and M are h-subhemi-rings of R and S respectively if and only if $\widetilde{\chi}_{L \rightrightarrows M} \in BSIH(R \cup S)$.

4.6 Definition

Let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ be a BS-set in $E = R \cup S$ over U and consider $\widetilde{\alpha}, \widetilde{\beta} \subseteq U$ with $\widetilde{\alpha} \cap \widetilde{\beta} = \emptyset$. Then

(i) The set $\widetilde{\gamma}_{\widetilde{\alpha}}^{\supseteq} = \{x \in R \mid \widetilde{\gamma}_R(x) \supseteq \widetilde{\alpha}\}$ is called upper $\widetilde{\alpha}$-inclusion of $\widetilde{B}$.

(ii) The set $\widetilde{\delta}_{\widetilde{\beta}}^{\supseteq} = \{\eta(a) \in S \mid \widetilde{\delta}_S(\eta(a)) \subseteq \widetilde{\beta} : a \in R\}$ is called upper $\widetilde{\beta}$-inclusion of $\widetilde{B}$.

(iii) The set $\widetilde{B}^{\supseteq}_{(\widetilde{\alpha},\widetilde{\beta})} = \left(\{x \in R \mid \widetilde{\gamma}_R(x) \supseteq \widetilde{\alpha}\}, \{\eta(x) \in S \mid \widetilde{\delta}_S(\eta(x)) \subseteq \widetilde{\beta} : a \in R\}\right)$ is called upper $\left(\widetilde{\alpha},\widetilde{\beta}\right)$-inclusion of $\widetilde{B}$.

4.7 Proposition

An $\left(\widetilde{\alpha},\widetilde{\beta}\right)$-inclusion $\widetilde{B}^{\supseteq}_{(\widetilde{\alpha},\widetilde{\beta})}$ of $\widetilde{B}$ a bipolar h-subhemi-ring (BhS-hemi-ring) of $R \cup S$ if and only if $\widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}}$ is a h-subhemi-ring of R and $\widetilde{\delta}^{\supseteq}_{\widetilde{\beta}}$ is a h-suhemi-ring S.

4.8 Theorem

If $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIH(R \cup S)$ and $\widetilde{\alpha} \in I_m(\widetilde{\gamma})$ and $\widetilde{\beta} \in I_m(\widetilde{\delta})$, then its $\left(\widetilde{\alpha},\widetilde{\beta}\right)$-inclusion $\widetilde{B}^{\supseteq}_{(\widetilde{\alpha},\widetilde{\beta})}$ is a BhS-hemi-ring of $R \cup S$ over U.

Proof. Let $\widetilde{B} \in BSIH(R \cup S)$ and $\widetilde{\alpha} \in I_m(\widetilde{\gamma}), \widetilde{\beta} \in I_m(\widetilde{\delta})$. For $a, b \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}}$, $\widetilde{\gamma}_R(a) \supseteq \widetilde{\alpha}, \widetilde{\gamma}_R(b) \supseteq \widetilde{\alpha}$ and $\widetilde{\delta}_S(\eta(a) \subseteq \widetilde{\beta}, \widetilde{\delta}_S(\eta(b) \subseteq \widetilde{\beta}$. Since $\widetilde{B} \in BSIH(R \cup S)$, we have $\widetilde{\gamma}_R(a+b) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b) \supseteq \widetilde{\alpha}$, $\widetilde{\delta}_S(\eta(a+b)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b)) \subseteq \widetilde{\beta}$ and $\widetilde{\gamma}_R(ab) \supseteq \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b) \supseteq \widetilde{\alpha}$, $\widetilde{\delta}_S(\eta(ab)) \subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}_S(\eta(b)) \subseteq \widetilde{\beta}$ this implies $b + c \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}}, \eta(b) + \eta(c) \in \widetilde{\delta}^{\supseteq}_{\widetilde{\beta}}$ and $bc \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}}, \eta(b)\eta(c) \in \widetilde{\delta}^{\supseteq}_{\widetilde{\beta}}$.

Let $c, d \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}}$ such that $x + c + z = d + z$ with $\widetilde{\gamma}_R(x) \supseteq \widetilde{\gamma}_R(c) \cap \widetilde{\gamma}_R(d) \supseteq \widetilde{\alpha} \Longrightarrow x \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}}$ and for $\eta(c), \eta(d) \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}}$ such that $\eta(x) + \eta(c) + \eta(z) = \eta(d) + \eta(z)$ with $\widetilde{\delta}_S(\eta(x)) \subseteq \widetilde{\delta}_S(\eta(c)) \cup \widetilde{\delta}_S(\eta(d)) \subseteq \widetilde{\beta} \Longrightarrow x \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}}, \eta(x) \in \widetilde{\delta}^{\supseteq}_{\widetilde{\beta}}$. Therefore $\widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}}$ and $\widetilde{\delta}^{\supseteq}_{\widetilde{\beta}}$ are h-subhemi-ring of R and S respectively. Hence $\widetilde{B}^{\supseteq}_{(\widetilde{\alpha},\widetilde{\beta})}$ is BhS-hemi-ring of $R \cup S$. □

4.9 Proposition

If $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ is a BS-set of $R \cup S$ over U, and $\left(\widetilde{\alpha},\widetilde{\beta}\right)$-inclusion is a BhS-hemi-ring of $R \cup S$ for each $\widetilde{\alpha}, \widetilde{\beta} \subseteq U$ and $BI_m(\widetilde{B})$ is an bipolar set by bipolar inclusion. Then $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIH(R \cup S)$.

Proof. Let $\left(\widetilde{\alpha},\widetilde{\beta}\right)$-inclusion be a BhS-hemi-ring of $R \cup S$ over U. Assume $a, b \in R$ be such that $\widetilde{\gamma}_R(a) = \widetilde{\alpha}', \widetilde{\gamma}_R(b) = \widetilde{\alpha}''$ and $\widetilde{\delta}_S(\eta(a) = \widetilde{\beta}', \widetilde{\delta}_S(\eta(b) = \widetilde{\beta}''$ where $\widetilde{\alpha}' \subseteq \widetilde{\alpha}''$, $\widetilde{\beta}' \supseteq \widetilde{\beta}''$ and $\widetilde{\alpha}', \widetilde{\alpha}'', \widetilde{\beta}', \widetilde{\beta}'' \subseteq U$. This implies $a \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}'}$, $b \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}''}$, $\eta(a) \in \widetilde{\delta}^{\supseteq}_{\widetilde{\beta}'}, \eta(b) \in \widetilde{\delta}^{\supseteq}_{\widetilde{\beta}''}$ and so $b \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}'}, \eta(b) \in \widetilde{\beta}^{\supseteq}_{\widetilde{\alpha}'}$. As $\widetilde{B}^{\supseteq}_{(\widetilde{\alpha},\widetilde{\beta})}$ is a BhS-hemi-ring in $R \cup S$ for all $\widetilde{\alpha}, \widetilde{\beta} \subseteq U$ so that $\widetilde{\gamma}_R(a+b) \supseteq \widetilde{\alpha}' = \widetilde{\alpha}' \cap \widetilde{\alpha}'' = \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b)$, $\widetilde{\delta}_S(\eta(a+b)) \subseteq \widetilde{\beta}' = \widetilde{\beta}' \cup \widetilde{\beta}'' = \widetilde{\delta}_S(\eta(a) \cup \widetilde{\delta}_S(\eta(b)$ and $\widetilde{\gamma}_R(ab) \supseteq \widetilde{\alpha}'_R = \widetilde{\alpha}'_R \cap \widetilde{\alpha}''_R = \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}_R(b)$, $\widetilde{\delta}_S(\eta(a)\eta(b)) \subseteq \widetilde{\beta}' = \widetilde{\beta}' \cup \widetilde{\beta}'' = \widetilde{\delta}_S(\eta(a) \cup \widetilde{\delta}_S(\eta(b)$.

Let $c \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}'}, d \in \widetilde{\gamma}^{\supseteq}_{\widetilde{\alpha}''}$ and $\eta(c) \in \widetilde{\delta}^{\supseteq}_{\widetilde{\beta}'}, \eta(d) \in \widetilde{\delta}^{\supseteq}_{\widetilde{\beta}''}$ where $\widetilde{\alpha}' \subseteq \widetilde{\alpha}''$ and $\widetilde{\beta}' \supseteq \widetilde{\beta}''$. Then $x + c + z = d + z$ with $\widetilde{\gamma}_R(x) \supseteq \widetilde{\alpha}' = \widetilde{\gamma}_R(c) \cap \widetilde{\gamma}_R(d)$ and $\eta(x) + \eta(c) + \eta(z) = \eta(d) + \eta(z)$ with $\widetilde{\delta}_S(\eta(x)) \subseteq \widetilde{\beta} = \widetilde{\beta}' \cup \widetilde{\beta}'' = \widetilde{\delta}_S(\eta(c) \cup \widetilde{\delta}_S(\eta(d)$.

Therefore, $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIH(R \cup S)$. □

4.10 Theorem

Let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ be a BS-set in $R \cup S$ over U, then $\widetilde{B} \in BSIH(R \cup S)$ if and only if it holds (B-Sh_3) and (B-Sh_4) $\widetilde{\gamma}_R \oplus_h \widetilde{\gamma}_R \sqsubseteq \widetilde{\gamma}_R$, $\widetilde{\delta}_S \oplus_h \widetilde{\delta}_S \sqsubseteq \widetilde{\delta}_S$ (B-Sh_5) $\widetilde{\gamma}_R \diamond_h \widetilde{\gamma}_R \sqsubseteq \widetilde{\gamma}_R$, $\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S \sqsubseteq \widetilde{\delta}_S$

Proof. Let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIH(R \cup S)$. If $(\widetilde{\gamma}_R \oplus_h \widetilde{\gamma}_R)(a) = \emptyset_R$ and $\left(\widetilde{\delta}_S \oplus_h \widetilde{\delta}_S\right)(\eta(a)) = U_S$ for $a \in R, \eta(a) \in S$, then $(\widetilde{\gamma}_R \oplus_h \widetilde{\gamma}_R)(a) \subseteq \widetilde{\gamma}_R(a)$, $\left(\widetilde{\delta}_S \oplus_h \widetilde{\delta}_S\right)(\eta(a)) \supseteq \widetilde{\delta}_S(\eta(a))$. Therefore, $\widetilde{\gamma}_R \oplus_h \widetilde{\gamma}_R \sqsubseteq \widetilde{\gamma}_R$, $\widetilde{\delta}_S \oplus_h \widetilde{\delta}_S \sqsubseteq \widetilde{\delta}_S$. Otherwise, let $a+x_1+y_1+b = x_2+y_2+b$ and $\eta(a)+\eta(x_1)+\eta(y_1)+\eta(b) = \eta(x_2)+\eta(y_2)+\eta(b)$ $\forall a, x_1, x_2, y_1, y_2, b \in R$ and $\eta(a), \eta(x_1), \eta(x_2), \eta(b), \eta(y_1), \eta(y_2) \in S$.

Then,

$$
\begin{aligned}
&(\widetilde{\gamma}_R \oplus_h \widetilde{\gamma}_R)(a)\\
&= \bigcup_{a+x_1+y_1+b=x_2+y_2+b} \{\widetilde{\gamma}_R(x_1) \cap \widetilde{\gamma}_R(x_2) \cap \widetilde{\gamma}_R(y_1) \cap \widetilde{\gamma}_R(y_2)\}\\
&\subseteq \bigcup_{a+x_1+y_1+b=x_2+y_2+b} \{\widetilde{\gamma}_R(x_1+y_1) \cap \widetilde{\gamma}_R(x_2+y_2)\}\\
&\subseteq \bigcup_{a+x_1+y_1+b=x_2+y_2+b} \{\widetilde{\gamma}_R(a)\}\\
&= \widetilde{\gamma}_R(a).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(\widetilde{\delta}_S \oplus_h \widetilde{\delta}_S)(a)\\
&= \bigcap_{\eta(a)+\eta(x_1)+\eta(y_1)+\eta(b)=\eta(x_2)+\eta(y_2)+\eta(b)} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(x_1)) \cup \widetilde{\delta}_S(\eta(x_2)) \\ \cup \widetilde{\delta}_S(\eta(y_1)) \cup \widetilde{\delta}_S(\eta(y_2)) \end{array} \right\}\\
&\supseteq \bigcap_{\eta(a)+\eta(x_1)+\eta(y_1)+\eta(b)=\eta(x_2)+\eta(y_2)+\eta(b)} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(x_1)+\eta(x_2)) \\ \cup \widetilde{\delta}_S(\eta(y_1)+\eta(y_2)) \end{array} \right\}\\
&\supseteq \bigcap_{\eta(a)+\eta(x_1)+\eta(y_1)+\eta(b)=\eta(x_2)+\eta(y_2)+\eta(b)} \left\{\widetilde{\delta}_S(\eta(a))\right\}\\
&= \widetilde{\delta}_S(\eta(a)).
\end{aligned}
$$

Therefore, $\widetilde{\gamma}_R \oplus_h \widetilde{\gamma}_R \widetilde{\subseteq} \widetilde{\gamma}_R$, $\widetilde{\delta}_S \oplus_h \widetilde{\delta}_S \widetilde{\supseteq} \widetilde{\delta}_S$.

Now as $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIH(R \cup S)$. If $(\widetilde{\gamma}_R \diamond_h \widetilde{\gamma}_R)(a) = \emptyset_R$ and $\left(\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S\right)(\eta(a)) = U_S$ for $a \in R, \eta(a) \in S$, then $(\widetilde{\gamma}_R \diamond_h \widetilde{\gamma}_R)(a) \subseteq \widetilde{\gamma}_R(a)$, $\left(\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S\right)(\eta(a)) \supseteq \widetilde{\delta}_S(\eta(a))$. Therefore, $\widetilde{\gamma}_R \diamond_h \widetilde{\gamma}_R \widetilde{\subseteq} \widetilde{\gamma}_R$, $\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S \widetilde{\supseteq} \widetilde{\delta}_S$. Otherwise, let $a + \sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b$ and $\eta(a) + \sum_{k=1}^{m} \eta(y_k)\eta(z_k) + \eta(b) = \sum_{k'=1}^{n} \eta(y'_{k'})\eta(z'_{k'}) + \eta(b)$ $\forall a, z_k, z'_{k'}, y_k, y'_{k'}, b \in R$ and $\eta(a), \eta(z_k), \eta(z'_{k'}), \eta(b), \eta(y_k), \eta(y'_{k'}) \in S$.

So that,

$$
\begin{aligned}
&(\widetilde{\gamma}_R \diamond_h \widetilde{\gamma}_R)(a)\\
&= \bigcup_{a+\sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b} \{\widetilde{\gamma}_R(y_k) \cap \widetilde{\gamma}_R(z_k) \cap \widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}_R(z'_{k'})\}\\
&\subseteq \bigcup_{a+\sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b} \{\widetilde{\gamma}_R(\sum_{k=1}^{m} y_k z_k) \cap \widetilde{\gamma}_R(\sum_{k'=1}^{n} y'_{k'} z'_{k'})\}\\
&\subseteq \bigcup_{a+\sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b} \{\widetilde{\gamma}_R(a)\}\\
&= \widetilde{\gamma}_R(a),\\
&(\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S)(a)
\end{aligned}
$$

$$= \bigcap_{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(b)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(b)} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(y_k)) \cup \widetilde{\delta}_S(\eta(z_k)) \\ \cup\widetilde{\delta}_S(\eta(y'_{k'})) \cup \widetilde{\delta}_S(\eta(z'_{k'})) \end{array} \right\}$$
$$\supseteq \bigcap_{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(b)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(b)} \left\{ \begin{array}{c} \widetilde{\delta}_S((\sum_{k=1}^{m}\eta(y_k)\eta(z_k))) \\ \cup\widetilde{\delta}_S((\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'}))) \end{array} \right\}$$
$$\supseteq \bigcap_{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(b)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(b)} \left\{ \widetilde{\delta}_S(\eta(a)) \right\}$$
$$= \widetilde{\delta}_S(\eta(a)).$$

Therefore, $\widetilde{\gamma}_R \diamond_h \widetilde{\gamma}_R \sqsubseteq \widetilde{\gamma}_R$, $\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S \sqsubseteq \widetilde{\delta}_S$.

Hence (B-Sh_4) and (B-Sh_5) hold.

Conversely, suppose (B-Sh_3),(B-Sh_4) and (B-Sh_5) hold.

$$\begin{aligned}
&\widetilde{\gamma}_R\,(a_1 + a_2) \\
&\supseteq (\widetilde{\gamma}_R \oplus_h \widetilde{\gamma}_R)\,(a_1 + a_2) \\
&= \bigcup_{a_1+a_2+x_1+y_1+b=x_2+y_2+b} \{\widetilde{\gamma}_R(x_1) \cap \widetilde{\gamma}_R(x_2) \cap \widetilde{\gamma}_R(y_1) \cap \widetilde{\gamma}_R(y_2)\} \\
&\supseteq \widetilde{\gamma}_R(a_1) \cap \widetilde{\gamma}_R(a_2) \cap \widetilde{\gamma}_R(0) \\
&= \widetilde{\gamma}_R(a_1) \cap \widetilde{\gamma}_R(a_2),
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\delta}_S(\eta(a_1) + \eta(a_2)) \\
&\subseteq (\widetilde{\delta}_S \oplus_h \widetilde{\delta}_S)(\eta(a_1) + \eta(a_2)) \\
&= \bigcap_{\eta(a_1)+\eta(a_2)+\eta(x_1)+\eta(y_1)+\eta(b)=\eta(x_2)+\eta(y_2)+\eta(b)} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(x_1)) \cup \widetilde{\delta}_S(\eta(x_2)) \\ \cup\widetilde{\delta}_S(\eta(y_1)) \cup \widetilde{\delta}_S(\eta(y_2)) \end{array} \right\} \\
&\subseteq \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(a_1)) \\ \cup\widetilde{\delta}_S(\eta(a_2)) \cup \widetilde{\delta}_S(\eta(0)) \end{array} \right\} \\
&= \widetilde{\delta}_S(\eta(a_1)) \cup \widetilde{\delta}_S(\eta(a_2))
\end{aligned}$$

Let $xa + \sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b$ and $\eta(xa) + \sum_{k=1}^{m}\eta(y_k)\eta(z_k) + \eta(b) = \sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'}) + \eta(b)$ $\forall x, a, z_k, z'_{k'}, y_k, y'_{k'}, b \in R$ and $\eta(x), \eta(a), \eta(z_k), \eta(z'_{k'}), \eta(b), \eta(y_k), \eta(y'_{k'}) \in S$.

So that,

$$\begin{aligned}
&\widetilde{\gamma}_R(a_1a_2) \\
&\supseteq (\widetilde{\gamma}_R \diamond_h \widetilde{\gamma}_R)(a_1a_2) \\
&= \bigcup_{a_1a_2+\sum_{k=1}^{m} y_k z_k+b=\sum_{k'=1}^{n} y'_{k'} z'_{k'}+b} \{\widetilde{\gamma}_R(y_k) \cap \widetilde{\gamma}_R(z_k) \cap \widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}_R(z'_{k'})\} \\
&\supseteq \bigcup_{a_1a_2+\sum_{k=1}^{m} y_k z_k+b_1=\sum_{k'=1}^{n} y'_{k'} z'_{k'}+b_1} \{\widetilde{\gamma}_R(\sum_{k=1}^{m} y_k z_k) \cap \widetilde{\gamma}_R(\sum_{k'=1}^{n} y'_{k'} z'_{k'})\} \\
&\supseteq \bigcup_{a_1a_2+\sum_{k=1}^{m} y_k z_k+b=\sum_{k'=1}^{n} y'_{k'} z'_{k'}+b} \{\widetilde{\gamma}_R(a_1) \cap \widetilde{\gamma}_R(a_2)\} \\
&= \widetilde{\gamma}_R(a_1) \cap \widetilde{\gamma}_R(a_2),
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\delta}_S(\eta(a_1)\eta(a_2))\\
&\subseteq (\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S)(\eta(a_1)\eta(a_2))\\
&= \bigcap_{\eta(a_1)\eta(a_2)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(b)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(b)} \left\{\begin{array}{c}\widetilde{\delta}_S(\eta(y_k))\cup\widetilde{\delta}_S(\eta(z_k))\\ \cup\widetilde{\delta}_S(\eta(y'_{k'}))\cup\widetilde{\delta}_S(\eta(z'_{k'}))\end{array}\right\}\\
&\subseteq \bigcap_{\eta(a_1)\eta(a_2)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(b_1)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(b_1)} \left\{\begin{array}{c}\widetilde{\delta}_S((\sum_{k=1}^{m}\eta(y_k)\eta(z_k)))\\ \cup\widetilde{\delta}_S((\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})))\end{array}\right\}\\
&\subseteq \bigcap_{\eta(a_1)\eta(a_2+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(b_1)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(b_1)} \left\{\widetilde{\delta}_S(\eta(a_1))\cup\widetilde{\delta}_S(\eta(a_2))\right\}\\
&= \widetilde{\delta}_S(\eta(a_1))\cup\widetilde{\delta}_S(\eta(a_2)).
\end{aligned}$$

Thus, (B-Sh_1) and (B-Sh_2) holds.

Therefore, $\widetilde{B} \in BSIH(R\cup S)$. □

5 BSI-h-ideals

In this section, we initiated BSI-h-ideals and presented some related properties.

5.1 Definition

A BS-set $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ over U is called a bipolar soft intersectional left(resp. right) h-ideal (briefly, BSI-left(resp. right) h-ideal) in union of two isomorphic hemi-rings R and S over U if it holds (B-Sh_1),(B-Sh_3), and

(B-Sh_6) $\widetilde{\gamma}_R(xy) \supseteq \widetilde{\gamma}_R(y)$, $\widetilde{\delta}_S(\eta(x)\eta(y)) \subseteq \widetilde{\delta}_S(\eta(y))$ (resp. $\widetilde{\gamma}_R(xy) \supseteq \widetilde{\gamma}_R(x)$, $\widetilde{\delta}_S(\eta(x)\eta(y)) \subseteq \widetilde{\delta}_S(\eta(x))$ for all $x, y \in R$.

A BS-set over U is called a BSI-h-ideal of $S\cup R$ if it is both a BSI-left h-ideal and a BSI-right h-ideal of $S \cup R$.

In rest of paper, set of all BSI-h-ideals (resp. BSI-left h-ideals, BSI-right h-ideal) of $R\cup S$ over U is denoted by $BSIhI(R\cup S)$ $(BSILhI(R\cup S), BSIRhI(R\cup S))$.

5.2 Example

Let $U = S_3$ symmetric-group, be a Universe. Let $R = \{0_R, a_R, b_R, c_R, d_R\}$ be a hemi-ring with zero "." and "+" defined as follows

$+$	0_R	a_R	b_R	c_R	d_R
0_R	0_R	a_R	b_R	c_R	d_R
a_R	a_R	a_R	d_R	d_R	d_R
b_R	b_R	d_R	d_R	d_R	d_R
c_R	c_R	d_R	d_R	d_R	d_R
d_R	d_R	d_R	d_R	d_R	d_R

Also let $R = \{0_S, 1_S, 2_S, 3_S, 4_S\}$ be a hemi-ring with zero "." and "$\oplus$" defined as follows

$\oplus$	0_S	1_S	2_S	3_S	4_S
0_S	0_S	1_S	2_S	3_S	4_S
1_S	1_S	1_S	4_S	4_S	4_S
2_S	2_S	4_S	4_S	4_S	4_S
3_S	3_S	4_S	4_S	4_S	4_S
4_S	4_S	4_S	4_S	4_S	4_S

Then $E = R \cup S$ and clearly $R \cap S = \emptyset$. We describe a BS-set $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ over U and there is an isomorphism $\eta : R \to S$ such that $\eta(0_R) = 0_S, \eta(a_R) = 1_S, \eta(b_R) = 2_S, \eta(c_R) = 3_S, \eta(d_R) = 4_S$. We have following table

	0_R	a_R	b_R	c_R	d_R
$\widetilde{\gamma}_R(x)$	S_3	$\{(12), (132)\}$	$\{(12), (132), (123)\}$	$\{(12), (132), (123)\}$	$\{(12), (132), (123)\}$
$\widetilde{\delta}_R(\eta(x))$	$\emptyset$	$\{(13)\}$	$\{(13), (23)\}$	$\{(13), (23)\}$	$\{(13), (23)\}$

Then $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIhI(R \cup S)$.

5.3 Proposition

If $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIhI(R \cup S)$, then $\widetilde{\gamma}_R(0_R) \supseteq \widetilde{\gamma}_R(x_R)$ and $\widetilde{\delta}_S(\eta(0_R)) \subseteq \widetilde{\delta}_S(\eta(x_R))$ $\forall 0_R, x_R \in R, \eta(0_R), \eta(x_R) \in S$.

5.4 Proposition

Two non-empty subsets L of R and M of S are h-ideals (resp. interior h-ideals) of R and S respectively if and only if BS-set $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ of $R \cup S$ over U is defined by

$$\widetilde{\gamma}_R(x) = \begin{cases} \alpha & \text{if } x \in R/L \\ \alpha' & \text{if } x \in L \end{cases}$$

$$\widetilde{\delta}_S(\eta(x)) = \begin{cases} \beta & \text{if } \eta(x) \in M \\ \beta' & \text{if } \eta(x) \in S/M \end{cases}$$

is a BSI-h-ideal (resp. $BSII$-h-ideal) of $R \cup S$ over U, with α,$\alpha', \beta, \beta' \subseteq U$ and $\alpha \subseteq \alpha'$, $\beta \subseteq \beta'$.

Proof. It is analogous to proof of Prop. 4.4. □

5.5 Corollary

Let $\emptyset \neq L_1$ and $\emptyset \neq L_2$ be two subset of E such that $L_1 \subseteq R, L_2 \subseteq S$. Then L_1 and L_2 are h-ideals of R and S respectively if and only if $\widetilde{\chi}_{L_1 \cup L_2} \in BSIhI(S \cup R)$.

5.6 Theorem

If $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in\in BSIhI(S \cup R)$ and $\widetilde{\alpha} \in I_m(\widetilde{\gamma})$ and $\widetilde{\beta} \in I_m(\widetilde{\delta})$, then its $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$-inclusion $\widetilde{B}^{\supseteq}_{(\widetilde{\alpha}, \widetilde{\beta})}$ is a BhS-ideal of $R \cup S$ over U.

Proof. It is analogous to proof of Theorem 4.8. □

5.7 Proposition

If $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ is a BS-set of $R \cup S$ over U, and $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$-inclusion is a BhS-h-ideal of $R \cup S$ for each $\widetilde{\alpha}, \widetilde{\beta} \subseteq U$ and $BI_m(\widetilde{B})$ is an bipolar set by bipolar inclusion. Then $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSIhH(R \cup S)$.

Proof. It is analogous to proof of Prop. 4.9. □

5.8 Theorem

Let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ be a BS-set of $R \cup S$, then $\widetilde{B} \in BSIhI(R \cup S)$ iff it holds (B-Sh_3), (B-Sh_4)and (B-Sh_7) $\widetilde{U}_R \diamond_h \widetilde{\gamma}_R \sqsubseteq \widetilde{\gamma}_R, \widetilde{\emptyset}_S \diamond_h \widetilde{\delta}_S \sqsubseteq \widetilde{\delta}_S \left(\widetilde{\gamma}_R \oplus_h \widetilde{U}_R \sqsubseteq \widetilde{\gamma}_R, \widetilde{\delta}_S \oplus_h \widetilde{\emptyset}_S \sqsubseteq \widetilde{\delta}_S\right)$.

Proof. Let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S) \in BSILhI(R \cup S)$. If $\left(\widetilde{U}_R \diamond_h \widetilde{\gamma}_R\right)(a) = \emptyset_R$ and $\left(\widetilde{\emptyset}_S \diamond_h \widetilde{\delta}_S\right)(\eta(a)) = U_S$ for $a \in R, \eta(a) \in S$, then $\left(\widetilde{U}_R \diamond_h \widetilde{\gamma}_R\right)(a) \subseteq \widetilde{\gamma}_R(a)$, $\left(\widetilde{\emptyset}_S \diamond_h \widetilde{\delta}_S\right)(\eta(a)) \supseteq \widetilde{\delta}_S(\eta(a))$. Therefore, $\widetilde{U}_R \diamond_h \widetilde{\gamma}_R \sqsubseteq \widetilde{\gamma}_R, \widetilde{\emptyset}_S \diamond_h \widetilde{\delta}_S \sqsubseteq \widetilde{\delta}_S$. Otherwise, let $a + \sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b$ and $\eta(a) + \sum_{k=1}^{m} \eta(y_k)\eta(z_k) + \eta(b) = \sum_{k'=1}^{n} \eta(y'_{k'})\eta(z'_{k'}) + \eta(b)$ $\forall a, z_k, z'_{k'}, y_k, y'_{k'}, b \in R$ and $\eta(a), \eta(z_k), \eta(z'_{k'}), \eta(b), \eta(y_k), \eta(y'_{k'}) \in S$.

So that,

$$
\begin{aligned}
&(\widetilde{U}_R \diamond_h \widetilde{\gamma}_R)(a)\\
&= \bigcup_{a + \sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b} \{\widetilde{U}_R(y_k) \cap \widetilde{U}_R(z_k) \cap \widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}_R(z'_{k'})\}\\
&\subseteq \bigcup_{a + \sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b} \{\widetilde{\gamma}_R(\sum_{k=1}^{m} y_k z_k) \cap \widetilde{\gamma}_R(\sum_{k'=1}^{n} y'_{k'} z'_{k'})\}\\
&\subseteq \bigcup_{a + \sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b} \{\widetilde{\gamma}_R(a)\}\\
&= \widetilde{\gamma}_R(a),\\
&(\widetilde{\emptyset}_S \diamond_h \widetilde{\delta}_S)(\eta(a))\\
&= \bigcap_{\eta(a) + \sum_{k=1}^{m} \eta(y_k)\eta(z_k) + \eta(b) = \sum_{k'=1}^{n} \eta(y'_{k'})\eta(z'_{k'}) + \eta(b)} \left\{ \begin{array}{c} \widetilde{\emptyset}_S(\eta(y_k)) \cup \widetilde{\emptyset}_S(\eta(z_k)) \\ \cup \widetilde{\delta}_S(\eta(y'_{k'})) \cup \widetilde{\delta}_S(\eta(z'_{k'})) \end{array} \right\}\\
&\supseteq \bigcap_{\eta(a) + \sum_{k=1}^{m} \eta(y_k)\eta(z_k) + \eta(b) = \sum_{k'=1}^{n} \eta(y'_{k'})\eta(z'_{k'}) + \eta(b)} \left\{ \begin{array}{c} \widetilde{\delta}_S((\sum_{k=1}^{m} \eta(y_k)\eta(z_k))) \\ \cup \widetilde{\delta}_S((\sum_{k'=1}^{n} \eta(y'_{k'})\eta(z'_{k'}))) \end{array} \right\}\\
&\supseteq \bigcap_{\eta(a) + \sum_{k=1}^{m} \eta(y_k)\eta(z_k) + \eta(b) = \sum_{k'=1}^{n} \eta(y'_{k'})\eta(z'_{k'}) + \eta(b)} \left\{\widetilde{\delta}_S(\eta(a))\right\}\\
&= \widetilde{\delta}_S(\eta(a)).
\end{aligned}
$$

Therefore, $\widetilde{U}_R \diamond_h \widetilde{\gamma}_R \sqsubseteq \widetilde{\gamma}_R$, $\widetilde{\emptyset}_S \diamond_h \widetilde{\delta}_S \sqsubseteq \widetilde{\delta}_S$. Likewise, we can obtain $\widetilde{\gamma}_R \oplus_h \widetilde{U}_R \sqsubseteq \widetilde{\gamma}_R, \widetilde{\delta}_S \oplus_h \widetilde{\emptyset}_S \sqsubseteq \widetilde{\delta}_S$.

Hence (B-Sh_7) holds.

Conversely, suppose (B-Sh_3),(B-Sh_4) and (B-Sh_7) hold.

Let $xa + \sum_{k=1}^{m} y_k z_k + b = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + b$ and $\eta(xa) + \sum_{k=1}^{m} \eta(y_k)\eta(z_k) + \eta(b) = \sum_{k'=1}^{n} \eta(y'_{k'})\eta(z'_{k'}) + \eta(b)$ $\forall x, a, z_k, z'_{k'}, y_k, y'_{k'}, b \in R$

and $\eta(x), \eta(a), \eta(z_k), \eta(z'_{k'}), \eta(b), \eta(y_k), \eta(y'_{k'}) \in S$.

So that,

$\widetilde{\gamma}_R(a_1 a_2)$

$\supseteq (\widetilde{U}_R \diamond_h \widetilde{\gamma}_R)(a_1 a_2)$

$= \bigcup_{a_1a_2+\sum_{k=1}^{m} y_kz_k+b=\sum_{k'=1}^{n} y'_{k'}z'_{k'}+b} \{\widetilde{U}_R(y_k) \cap \widetilde{U}_R(z_k) \cap \widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}_R(z'_{k'})\}$
$\supseteq \widetilde{U}_R(a_1) \cap \widetilde{\gamma}_R(a_2),$
$= \widetilde{\gamma}_R(a_2),$
$\widetilde{\delta}_S(\eta(a_1)\eta(a_2))$
$\subseteq (\widetilde{\delta}_S \diamond_h \widetilde{\delta}_S)(\eta(a_1)\eta(a_2))$
$= \bigcap_{\eta(a_1)\eta(a_2)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(b)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(b)} \left\{ \begin{array}{c} \widetilde{\emptyset}_S(\eta(y_k)) \cup \widetilde{\emptyset}_S(\eta(z_k)) \\ \cup\widetilde{\delta}_S(\eta(y'_{k'})) \cup \widetilde{\delta}_S(\eta(z'_{k'})) \end{array} \right\}$
$\subseteq \widetilde{\emptyset}_S(\eta(a_1)) \cup \widetilde{\delta}_S(\eta(a_2))$
$= \widetilde{\delta}_S(\eta(a_2)).$

In the same manner, we can obtain $\widetilde{\gamma}_R(a_1a_2) \supseteq \widetilde{\gamma}_R(a_1)$, $\widetilde{\delta}_S(\eta(a_1)\eta(a_2)) \subseteq \widetilde{\delta}_S(\eta(a_2))$.

Thus, (B-Sh_6) holds. Hence $\widetilde{B} \in BSIhI(R \cup S)$. □

5.9 Proposition

If $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ and $\widetilde{B}^* = (\widetilde{\gamma}^*_R, \widetilde{\delta}^*_S) \in BSIhI(R \cup S)$, then $\widetilde{B}\widetilde{\cap}\widetilde{B}^* \in BSIhI(R \cup S)$.

Proof. Assume $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ and $\widetilde{B}^* = (\widetilde{\gamma}^*_R, \widetilde{\delta}^*_S)$ be two BSI-left h-ideals of $R \cup S$. Then for $a, b \in R$ with $\eta(a), \eta(b) \in S$

$\left(\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}^*_R\right)(a+b)$
$= \widetilde{\gamma}_R\,(a+b) \cap \widetilde{\gamma}^*_R\,(a+b)$
$\supseteq \widetilde{\gamma}_R\,(a) \cap \widetilde{\gamma}_R\,(b) \cap \widetilde{\gamma}^*_R\,(a) \cap \widetilde{\gamma}^*_R\,(b)$
$= \left(\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}^*_R\right)(a) \cap \left(\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}^*_R\right)(b)$
$\left(\widetilde{\delta}_R\widetilde{\cap}\widetilde{\delta}^*_R\right)(\eta(a)+\eta(b))$
$= \widetilde{\delta}_R\,(\eta(a)+\eta(b)) \cup \widetilde{\delta}^*_R\,(\eta(a)+\eta(b))$
$\subseteq \widetilde{\delta}_R\,(\eta(a)) \cup \widetilde{\delta}_R\,(\eta(b)) \cup \widetilde{\delta}^*_R\,(\eta(a)) \cup \widetilde{\delta}^*_R\,(\eta(b))$
$= \left(\widetilde{\delta}_R\widetilde{\cap}\widetilde{\delta}^*_R\right)(\eta(a)) \cup \left(\widetilde{\delta}_R\widetilde{\cap}\widetilde{\delta}^*_R\right)(\eta(b))$
Now for $x \in R$, $b \in R$ with $\eta(x), \eta(b) \in S$
$\left(\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}^*_R\right)(xb)$
$= \widetilde{\gamma}_R\,(xb) \cap \widetilde{\gamma}^*_R\,(xb)$
$\supseteq \widetilde{\gamma}_R\,(b) \cap \widetilde{\gamma}^*_R\,(b)$
$= \left(\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}^*_R\right)(b)$
$\left(\widetilde{\delta}_R\widetilde{\cap}\widetilde{\delta}^*_R\right)(\eta(a)\eta(b))$
$= \widetilde{\delta}_R\,(\eta(a)\eta(b)) \cup \widetilde{\delta}^*_R\,(\eta(a)\eta(b))$
$\subseteq \widetilde{\delta}_R\,(\eta(b)) \cup \widetilde{\delta}^*_R\,(\eta(b))$
$= \left(\widetilde{\delta}_R\widetilde{\cap}\widetilde{\delta}^*_R\right)(\eta(b))$

Let $c, x, d, z \in R, \eta(c), \eta(x), \eta(d), \eta(z) \in S$ with $x+c+z = d+z$ and $\eta(x)+\eta(c)+\eta(z) = \eta(d)+\eta(z)$. Then

$\left(\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}^*_R\right)(x)$
$= \widetilde{\gamma}_R\,(x) \cap \widetilde{\gamma}^*_R\,(x)$
$\supseteq \widetilde{\gamma}_R\,(c) \cap \widetilde{\gamma}_R\,(d) \cap \widetilde{\gamma}^*_R\,(c) \cap \widetilde{\gamma}^*_R\,(d)$
$= \left(\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}^*_R\right)(c) \cap \left(\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}^*_R\right)(d)$

$$\begin{aligned}
&\left(\widetilde{\delta}_R \widetilde{\cap} \widetilde{\delta}^*_R\right)(\eta(x)) \\
&= \widetilde{\delta}_R(\eta(x)) \cup \widetilde{\delta}^*_R(\eta(x)) \\
&\subseteq \widetilde{\delta}_R(\eta(c)) \cup \widetilde{\delta}_R(\eta(d)) \cup \widetilde{\delta}^*_R(\eta(c)) \cup \widetilde{\delta}^*_R(\eta(d)) \\
&= \left(\widetilde{\delta}_R \widetilde{\cap} \widetilde{\delta}^*_R\right)(\eta(c)) \cup \left(\widetilde{\delta}_R \widetilde{\cap} \widetilde{\delta}^*_R\right)(\eta(d)).
\end{aligned}$$

Hence $\widetilde{B} \cap \widetilde{B}^* \in BSILhI(R \cup S)$. In the like manner, we can obtain $\widetilde{B} \cap \widetilde{B}^* \in BSIRhI(R \cup S)$. □

5.10 Definition

Let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ and $\widetilde{B}^* = (\widetilde{\gamma}^*_R, \widetilde{\delta}^*_S)$ be two BS-set of $R \cup S$ over U. Then the cartesian product $\widetilde{B}$ and $\widetilde{B}^*$ is defined by

$$(\widetilde{B} \times \widetilde{B}^*)(x) = ((\widetilde{\gamma}_R \times \widetilde{\gamma}^*_R)(x), (\widetilde{\delta}_S \times \widetilde{\delta}^*_S)(\eta(x))),$$

where $(\widetilde{\gamma}_R \times \widetilde{\gamma}^*_R)(a, b) = \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}^*_R(b)$, $(\widetilde{\delta}_S \times \widetilde{\delta}^*_S)(\eta(a), \eta(b)) = \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}^*_S(\eta(b))\ \forall\, a, b \in R, \eta(a), \eta(b) \in S$.

5.11 Theorem

Let $\widetilde{B} = (\widetilde{\gamma}_{R_1}, \widetilde{\delta}_{S_1})$ and $\widetilde{B}^* = (\widetilde{\gamma}^*_{R_2}, \widetilde{\delta}^*_{S_2})$ be two BSI-h-ideals of $R_1 \cup S_1$ and $R_2 \cup S_2$ respectively and $R_1 \cap S_1 = R_2 \cap S_2 = \emptyset$ with isomorphic mappings $\eta : R_1 \longrightarrow S_1$ and $\eta' : R_2 \longrightarrow S_2$. Then $\widetilde{B} \times \widetilde{B}^*$ is a BSI-ideal of $(R_1 \times R_2) \cup (S_1 \times S_2)$ over U.

Proof. Let $\widetilde{B} = (\widetilde{\gamma}_{R_1}, \widetilde{\delta}_{S_1}) \in BSILhI(R_1 \cup S_1)$ and $\widetilde{B}^* = (\widetilde{\gamma}^*_{R_2}, \widetilde{\delta}^*_{S_2}) \in BSILhI(R_2 \cup S_2)$. Suppose $(a_1, b_1), (a_2, b_2) \in R_1 \times R_2$, with $(\eta(a_1), \eta'(b_1)), (\eta(a_2), \eta'(b_2)) \in S_1 \times S_2$. Then

$$\begin{aligned}
&(\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})((a_1, b_1) + (a_2, b_2)) \\
&= (\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})(a_1 + a_2, b_1 + b_2) \\
&= \widetilde{\gamma}_{R_1}(a_1 + a_2) \cap \widetilde{\gamma}^*_{R_2}(b_1 + b_2) \\
&\supseteq \widetilde{\gamma}_{R_1}(a_1) \cap \widetilde{\gamma}_{R_2}(a_2) \cap \widetilde{\gamma}^*_{R_1}(b_1) \cap \widetilde{\gamma}^*_{R_2}(b_2) \\
&= ((\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})(a_1, b_1)) \cap ((\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})(a_2, b_2)), \\
&(\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})((\eta(a_1), \eta'(b_1)) + (\eta(a_2), \eta'(b_2))) \\
&= (\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})(\eta(a_1) + \eta(a_2), \eta'(b_1) + \eta'(b_2)) \\
&= \widetilde{\delta}_{S_1}(\eta(a_1) + \eta(a_2)) \cup \widetilde{\delta}^*_{S_2}(\eta'(b_1) + \eta'(b_2)) \\
&\subseteq \widetilde{\delta}_{S_1}(\eta(a_1)) \cup \widetilde{\delta}^*_{S_2}(\eta'(b_1)) \cup \widetilde{\delta}_{S_1}(\eta(a_2)) \cup \widetilde{\delta}^*_{S_2}(\eta'(b_2)) \\
&= ((\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})(\eta(a_1), \eta'(b_1))) \cup ((\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})(\eta(a_2), \eta(b_2))).
\end{aligned}$$

Since $\widetilde{B} = (\widetilde{\gamma}_{R_1}, \widetilde{\delta}_{S_1}) \in BSILhI(R_1 \cup S_1)$ and $\widetilde{B}^* = (\widetilde{\gamma}^*_{R_2}, \widetilde{\delta}^*_{S_2}) \in BSILhI(R_2 \cup S_2)$.

So that

$$\begin{aligned}
&(\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})((a_1, b_1)(a_2, b_2)) \\
&= (\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})(a_1 a_2, b_1 b_2) \\
&= \widetilde{\gamma}_{R_1}(a_1 a_2) \cap \widetilde{\gamma}^*_{R_2}(b_1 b_2) \\
&\supseteq \widetilde{\gamma}_{R_1}(a_2) \cap \widetilde{\gamma}^*_{R_2}(b_2) \\
&= ((\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})(a_2, b_2)) \\
&(\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})((\eta(a_1), \eta'(b_1))(\eta(a_2), \eta'(b_2))) \\
&= (\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})(\eta(a_1)\eta(a_2), \eta'(b_1)\eta'(b_2)) \\
&= \widetilde{\delta}_{S_1}(\eta(a_1)\eta(a_2)) \cup \widetilde{\delta}^*_{S_2}(\eta'(b_1)\eta'(b_2))
\end{aligned}$$

$\subseteq \widetilde{\delta}_{S_1}(\eta(a_2)) \cup \widetilde{\delta}^*_{S_2}(\eta'(b_2))$
$= ((\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})(\eta(a_2), \eta'(b_2)))$
Now consider $(x, y) + (a_1, b_1) + (x_1, x_2) = (a_2, b_2) + (x_1, x_2)$ with
$(\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})(x, y)$
$= \widetilde{\gamma}_{R_1}(x) \cap \widetilde{\gamma}^*_{R_2}(y)$
$\supseteq (\widetilde{\gamma}_{R_1}(a_1) \cap \widetilde{\gamma}^*_{R_2}(b_1)) \cap (\widetilde{\gamma}_{R_1}(a_2) \cap \widetilde{\gamma}^*_{R_2}(b_2))$
$= ((\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})(a_1, b_1)) \cap ((\widetilde{\gamma}_{R_1} \times \widetilde{\gamma}^*_{R_2})(a_2, b_2))$
and $(\eta(x), \eta(y)) + (\eta(a_1), \eta(b_1)) + (\eta(x_1), \eta(x_2)) = (\eta(a_2), \eta(b_2)) + (\eta(x_1), \eta(x_2))$
$(\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})(\eta(x), \eta'(y))$
$= \widetilde{\delta}_{S_1}(\eta(x)) \cup \widetilde{\delta}^*_{S_2}(\eta'(y))$
$\subseteq (\widetilde{\delta}_{S_1}(\eta(a_1)) \cup \widetilde{\delta}^*_{S_2}(\eta'(b_1))) \cup (\widetilde{\delta}_{S_1}(\eta(a_2)) \cup \widetilde{\delta}^*_{S_2}(\eta'(b_2)))$
$= ((\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})(\eta(a_1), \eta'(b_1))) \cup ((\widetilde{\delta}_{S_1} \times \widetilde{\delta}^*_{S_2})(\eta(a_2), \eta'(b_2)))$
Hence $\widetilde{B} \times \widetilde{B}^*$ is a BSI-h-ideal of $(R_1 \times R_2) \cup (S_1 \times S_2)$. □

5.12 Theorem

Let $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$,$\widetilde{B}^* = (\widetilde{\gamma}^*_R, \widetilde{\delta}^*_S) \in BSIhI(R \cup S)$. Then $\widetilde{B} \times \widetilde{B}^*$ is an BSI-h-ideal of $(R \times R) \cup (S \times S)$ over U.

Proof. It is analogous to proof of Theorem 5.11 □

5.13 Proposition

If $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$, $\widetilde{B}^* = (\widetilde{\gamma}^*_R, \widetilde{\delta}^*_S) \in BSIhI(R \cup S)$, then $\widetilde{B} \diamond \widetilde{B}^* \in BSIhI(R \cup S)$.

Proof. Assume $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ and $\widetilde{B}^* = (\widetilde{\gamma}^*_R, \widetilde{\delta}^*_S)$ be two BSI-left h-ideal of $R \cup S$ and $a, b \in R, \eta(a), \eta(b) \in S$. Then

(i) $(\widetilde{\gamma}_R \diamond \widetilde{\gamma}^*_R)(a) \cap (\widetilde{\gamma}_R \diamond \widetilde{\gamma}^*_R)(b)$

$$= \left\{ \bigcup_{a + \sum_{k=1}^{m} y_k z_k + c = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + c} \left\{ \begin{array}{c} \widetilde{\gamma}_R(y_k) \cap \widetilde{\gamma}_R(z_k) \\ \cap \widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}_R(z'_{k'}) \end{array} \right\} \right\}$$

$$\cap \left\{ \bigcup_{b + \sum_{k=1}^{m'} s_k t_k + d = \sum_{k'=1}^{n'} s'_{k'} t'_{k'} + d} \left\{ \begin{array}{c} \widetilde{\gamma}_R(s_k) \cap \widetilde{\gamma}_R(t_k) \\ \cap \widetilde{\gamma}_R(s'_{k'}) \cap \widetilde{\gamma}_R(t'_{k'}) \end{array} \right\} \right\}$$

$$= \bigcup_{a + \sum_{k=1}^{m} y_k z_k + c = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + c} \; \bigcup_{b + \sum_{k=1}^{m'} s_k t_k + d = \sum_{k'=1}^{n'} s'_{k'} t'_{k'} + d} \left\{ \begin{array}{c} \widetilde{\gamma}_R(y_k) \cap \widetilde{\gamma}_R(z_k) \\ \cap \widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}_R(z'_{k'}) \\ \cap \widetilde{\gamma}_R(s_k) \cap \widetilde{\gamma}_R(t_k) \\ \cap \widetilde{\gamma}_R(s'_{k'}) \cap \widetilde{\gamma}_R(t'_{k'}) \end{array} \right\}$$

$$\subseteq \bigcup_{a + b + \sum_{k=1}^{p} e_k f_k + c + d = \sum_{k'=1}^{q} e'_{k'} f'_{k'} + c + d} \left\{ \begin{array}{c} \widetilde{\gamma}_R(e_k) \cap \widetilde{\gamma}_R(f_k) \\ \cap \widetilde{\gamma}_R(e'_{k'}) \cap \widetilde{\gamma}_R(f'_{k'}) \end{array} \right\}$$

$p = \max\{m, m'\}$ and $q = \max\{n, n'\}$
$e_k f_k = y_k z_k + s_k t_k, e'_{k'} f'_{k'} = y'_{k'} z'_{k'} + s'_{k'} t'_{k'}$
$= (\widetilde{\gamma}_R \diamond \widetilde{\gamma}^*_R)(a + b)$
$\left(\widetilde{\delta}_S \diamond \widetilde{\delta}^*_S\right)(\eta(a)) \cup \left(\widetilde{\delta}_S \diamond \widetilde{\delta}^*_S\right)(\eta(b))$

$$= \left\{ \bigcap_{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(c)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(c)} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(y_k)) \cup \widetilde{\delta}_S(\eta(z_k)) \\ \cup\widetilde{\delta}_S(\eta(y'_{k'})) \cup \widetilde{\delta}_S(\eta(z'_{k'})) \end{array} \right\} \right\}$$

$$\cup \left\{ \bigcap_{\eta b+\sum_{k=1}^{m'}\eta s_k \eta t_k+\eta d=\sum_{k'=1}^{n'}\eta s'_{k'} \eta t'_{k'}+\eta d} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(s_k)) \cup \widetilde{\delta}_S(\eta(t_k)) \\ \cup\widetilde{\delta}_S(\eta(s'_{k'})) \cup \widetilde{\delta}_S(\eta(t'_{k'})) \end{array} \right\} \right\}$$

$$= \bigcap_{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(c)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(c)} \bigcap_{\eta b+\sum_{k=1}^{m'}\eta s_k \eta t_k+\eta d=\sum_{k'=1}^{n'}\eta s'_{k'} \eta t'_{k'}+\eta d} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(y_k)) \cup \widetilde{\delta}_S(\eta(z_k)) \\ \cup\widetilde{\delta}_S(\eta(y'_{k'})) \cup \widetilde{\delta}_S(\eta(z'_{k'})) \\ \cup\widetilde{\delta}_S(\eta(s_k)) \cup \widetilde{\delta}_S(\eta(t_k)) \\ \cup\widetilde{\delta}_S(\eta(s'_{k'})) \cup \widetilde{\delta}_S(\eta(t'_{k'})) \end{array} \right\}$$

$$\supseteq \bigcap_{\substack{\eta(a)+\eta(b)+\sum_{k=1}^{p}\eta(e_k)\eta(f_k) \\ +\eta(c)+\eta(d)=\sum_{k'=1}^{q}\eta(e'_{k'})\eta(f'_{k'})+\eta(c)+\eta(d)}} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(e_k)) \cup \widetilde{\delta}_S(\eta(f_k)) \\ \cup\widetilde{\delta}_S(\eta(e'_{k'})) \cup \widetilde{\delta}_S(\eta(f'_{k'})) \end{array} \right\}$$

$p = \max\{m, m'\}$ and $q = \max\{n, n'\}$

$\eta(e_k f_k) = \eta(y_k z_k) + \eta(s_k t_k), \eta(e'_{k'} f'_{k'}) = \eta(y'_{k'} z'_{k'}) + \eta(s'_{k'} t'_{k'})$

$= \left(\widetilde{\delta}_S \diamond \widetilde{\delta}_S^*\right)(\eta(a) + \eta(b))$

$(\widetilde{\gamma}_R \diamond \widetilde{\gamma}_R^*)(a)$

$$= \left\{ \bigcup_{a+\sum_{k=1}^{m} y_k z_k+c=\sum_{k'=1}^{n} y'_{k'} z'_{k'}+c} \left\{ \begin{array}{c} \widetilde{\gamma}_R(y_k) \cap \widetilde{\gamma}_R(z_k) \\ \cap\widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}_R(z'_{k'}) \end{array} \right\} \right\}$$

$$\subseteq \bigcup_{ab+\sum_{k=1}^{m} y_k (z_k b)+cb=\sum_{k'=1}^{n} y'_{k'} (z'_{k'} b)+cb} \{\widetilde{\gamma}_R(y_k) \cap \widetilde{\gamma}_R(z_k b) \cap \widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}_R(z'_{k'} b)\}$$

$= (\widetilde{\gamma}_R \diamond \widetilde{\gamma}_R^*)(ab)$

$\left(\widetilde{\delta}_S \diamond \widetilde{\delta}_S^*\right)(\eta(a))$

$$= \left\{ \bigcap_{\substack{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(c) \\ =\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(c)}} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(y_k)) \cup \widetilde{\delta}_S(\eta(z_k)) \\ \cup\widetilde{\delta}_S(\eta(y'_{k'})) \cup \widetilde{\delta}_S(\eta(z'_{k'})) \end{array} \right\} \right\}$$

$$\supseteq \bigcap_{\substack{\eta(a)\eta(b)+\sum_{k=1}^{m}\eta(y_k)(\eta(z_k b))+\eta(cb) \\ =\sum_{k'=1}^{n}\eta(y'_{k'})(\eta(z'_{k'} b))+\eta(cb)}} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(e_k)) \cup \widetilde{\delta}_S(\eta(f_k)) \cup \\ \widetilde{\delta}_S(\eta(e'_{k'})) \cup \widetilde{\delta}_S(\eta(f'_{k'})) \end{array} \right\}$$

$= \left(\widetilde{\delta}_S \diamond \widetilde{\delta}_S^*\right)(\eta(a)\eta(b))$

In the like manner, for $c, x, d, z \in R, \eta(c), \eta(x), \eta(d), \eta(z) \in S$ with $x + c + z = d + z$ and

$\eta(x)+\eta(c)+\eta(z)=\eta(d)+\eta(z)$. Then,

$$\left(\widetilde{\gamma}_R\diamond\widetilde{\gamma}_R^*\right)(a)\cap\left(\widetilde{\gamma}_R\diamond\widetilde{\gamma}_R^*\right)(b)\subseteq\left(\widetilde{\gamma}_R\diamond\widetilde{\gamma}_R^*\right)(a+b),$$
$$\left(\widetilde{\delta}_S\diamond\widetilde{\delta}_S^*\right)(\eta(a))\cup\left(\widetilde{\delta}_S\diamond\widetilde{\delta}_S^*\right)(\eta(b))\supseteq\left(\widetilde{\delta}_S\diamond\widetilde{\delta}_S^*\right)(\eta(a)+\eta(b)).$$

Then $\widetilde{B}\diamond\,\widetilde{B}^*\in BSILhI(R\cup S)$. Likewise, we can obtain $\widetilde{B}\diamond\,\widetilde{B}^*\in BSIRhI(R\cup S)$. □

6 h-Hemi-Regular Hemi-Rings

In this section, we discussed some characterizations of h-hemi-regular hemi-rings by bipolar soft int. h-ideals.

6.1 Definition[31]

A hemi-ring S is called h-hemi-regular if for each $a\in S$, there exist x_1 ,x_2 , $b\in S$ such that $a+ax_1a+b=ax_2a+b$.

6.2 Lemma[31]

For any right h-ideal L_1 and any left h-ideal L_2 of S, then $L_1\cap L_2\supseteq\overline{L_1L_2}$.

6.3 Lemma[31]

For any right h-ideal L_1 and any left h-ideal L_2 of S, we have $L_1\cap L_2=\overline{L_1L_2}$ if and only if S is hemi-regular hemi-ring.

6.4 Lemma

Let $\widetilde{B}=(\widetilde{\gamma}_R,\widetilde{\delta}_S)$ be a BSI-left h-ideal and $\widetilde{B}^*=(\widetilde{\gamma}_R^*,\widetilde{\delta}_S^*)$ be a BSI-right h-ideal of $R\cup S$ over U, then $\widetilde{\gamma}_R\diamond\widetilde{\gamma}_R^*\sqsubseteq\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}_R^*$ and $\widetilde{\delta}_S\diamond\widetilde{\delta}_S^*\sqsubseteq\widetilde{\delta}_S\widetilde{\cup}\widetilde{\delta}_S^*$.

Proof. Assume $\widetilde{\gamma}_R\diamond\widetilde{\gamma}_R^*=\widetilde{\emptyset}_R$ and $\widetilde{\delta}_S\diamond\widetilde{\delta}_S^*\widetilde{\supseteq}\widetilde{U}_S$, then clearly $\widetilde{\gamma}_R\diamond\widetilde{\gamma}_R^*\sqsubseteq\widetilde{\gamma}_R\widetilde{\cap}\widetilde{\gamma}_R^*$ and $\widetilde{\delta}_S\diamond\widetilde{\delta}_S^*\sqsubseteq\widetilde{\delta}_S\widetilde{\cup}\widetilde{\delta}_S^*$. Otherwise, for $a,b\in R$ $\eta(a),\eta(b)\in S$

$$\begin{aligned}
&\left(\widetilde{\gamma}_R\diamond\widetilde{\gamma}_R^*\right)(a)\\
&=\bigcup_{a+\sum\limits_{k=1}^{m}y_kz_k+c=\sum\limits_{k'=1}^{n}y'_{k'}z'_{k'}+c}\left\{\begin{array}{c}\widetilde{\gamma}_R(y_k)\cap\widetilde{\gamma}_R^*(z_k)\\ \cap\widetilde{\gamma}_R(y'_{k'})\cap\widetilde{\gamma}_R^*(z'_{k'})\end{array}\right\}\\
&\subseteq\bigcup_{a+\sum\limits_{k=1}^{m}y_kz_k+c=\sum\limits_{k'=1}^{n}y'_{k'}z'_{k'}+c}\left\{\begin{array}{c}\widetilde{\gamma}_R(\sum\limits_{k=1}^{m}y_kz_k)\cap\widetilde{\gamma}_R^*(\sum\limits_{k'=1}^{n}y'_{k'}z'_{k'})\\ \cap\widetilde{\gamma}_R(\sum\limits_{k=1}^{m}y_kz_k)\cap\widetilde{\gamma}_R^*(\sum\limits_{k'=1}^{n}y'_{k'}z'_{k'})\end{array}\right\}\\
&\subseteq\bigcup_{a+\sum\limits_{k=1}^{m}y_kz_k+c=\sum\limits_{k'=1}^{n}y'_{k'}z'_{k'}+c}\{\widetilde{\gamma}_R(a)\cap\widetilde{\gamma}_R^*(a)\}\\
&=\left(\widetilde{\gamma}_R\cap\widetilde{\gamma}_R^*\right)(a)\\
&\left(\widetilde{\delta}_S\diamond\widetilde{\delta}_S^*\right)(\eta(a))\\
&=\bigcap_{\eta(a)+\sum\limits_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(c)=\sum\limits_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(c)}\left\{\begin{array}{c}\widetilde{\delta}_S(\eta(y_k))\cup\widetilde{\delta}_S^*(\eta(z_k))\\ \cup\widetilde{\delta}_S(\eta(y'_{k'}))\cup\widetilde{\delta}_S^*(\eta(z'_{k'}))\end{array}\right\}
\end{aligned}$$

$$\supseteq \bigcap_{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(c) = \sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(c)} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(\sum_{k=1}^{m} y_k z_k)) \cup \widetilde{\delta}^*_S(\eta(\sum_{k'=1}^{n} y'_{k'} z'_{k'})) \\ \cup\widetilde{\delta}_S(\eta(\sum_{k=1}^{m} y_k z_k)) \cup \widetilde{\delta}^*_S(\eta(\sum_{k'=1}^{n} y'_{k'} z'_{k'})) \end{array} \right\}$$

$$\supseteq \bigcap_{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(c)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(c)} \left\{ \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}^*_S(\eta(a)) \right\}$$

$$= \left(\widetilde{\delta}_S \cup \widetilde{\delta}^*_S\right)(\eta(a))$$

Hence $\widetilde{\gamma}_R \diamond \widetilde{\gamma}^*_R \sqsubseteq \widetilde{\gamma}_R \widetilde{\cap} \widetilde{\gamma}^*_R$ and $\widetilde{\delta}_S \diamond \widetilde{\delta}^*_S \sqsubseteq \widetilde{\delta}_S \widetilde{\cup} \widetilde{\delta}^*_S$. □

6.5 Theorem

Following conditions are equivalent, for any two hemi-rings R and S with $R \cap S = \emptyset$:

(i) Both R and S are hemi-regular

(ii) $\widetilde{\gamma}_R \diamond \widetilde{\gamma}^*_R = \widetilde{\gamma}_R \widetilde{\cap} \widetilde{\gamma}^*_R$ and $\widetilde{\delta}_S \diamond \widetilde{\delta}^*_S = \widetilde{\delta}_S \widetilde{\cup} \widetilde{\delta}^*_S$ for any BSI-left h-ideal $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ and BSI-right h-ideal $\widetilde{B}^* = (\widetilde{\gamma}^*_R, \widetilde{\delta}^*_S)$ of $R \cup S$ over U.

Proof. (i)⟹(ii) Let R and S be two hemi-regular hemi-ring and $\widetilde{B} = (\widetilde{\gamma}_R, \widetilde{\delta}_S)$ be a BSI-left h-ideal and $\widetilde{B}^* = (\widetilde{\gamma}^*_R, \widetilde{\delta}^*_S)$ be a BSI-right h-ideal of $R \cup S$ over U. There exist $a \in R, \eta(a) \in S$, for x_1 ,x_2 , $b \in R, \eta(x_1)$,$\eta(x_2)$, $\eta(b) \in S$ such that $a + ax_1a + b = ax_2a + b$ and $\eta(a) + \eta(a)\eta(x_1)\eta(a) + \eta(b) = \eta(a)\eta(x_2)\eta(a) + \eta(b)$. Then

$$(\widetilde{\gamma}_R \diamond \widetilde{\gamma}^*_R)(a)$$

$$= \bigcup_{a+\sum_{k=1}^{m} y_k z_k + c = \sum_{k'=1}^{n} y'_{k'} z'_{k'} + c} \left\{ \begin{array}{c} \widetilde{\gamma}_R(y_k) \cap \widetilde{\gamma}^*_R(z_k) \\ \cap\widetilde{\gamma}_R(y'_{k'}) \cap \widetilde{\gamma}^*_R(z'_{k'}) \end{array} \right\}$$

$$\supseteq \widetilde{\gamma}_R(ay_k) \cap \widetilde{\gamma}^*_R(ay'_{k'}) \cap \widetilde{\gamma}_R(a) \cap \widetilde{\gamma}^*_R(a)$$

$$\supseteq (\widetilde{\gamma}_R)(a) \cap (\widetilde{\gamma}^*_R)(a)$$

$$= (\widetilde{\gamma}_R \cap \widetilde{\gamma}^*_R)(a)$$

$$\left(\widetilde{\delta}_S \diamond \widetilde{\delta}^*_S\right)(\eta(a))$$

$$= \bigcap_{\eta(a)+\sum_{k=1}^{m}\eta(y_k)\eta(z_k)+\eta(c)=\sum_{k'=1}^{n}\eta(y'_{k'})\eta(z'_{k'})+\eta(c)} \left\{ \begin{array}{c} \widetilde{\delta}_S(\eta(y_k)) \cup \widetilde{\delta}^*_S(\eta(z_k)) \\ \cup\widetilde{\delta}_S(\eta(y'_{k'})) \cup \widetilde{\delta}^*_S(\eta(z'_{k'})) \end{array} \right\}$$

$$\subseteq \widetilde{\delta}_S(\eta(a)\eta(y_k)) \cup \widetilde{\delta}^*_S(\eta(a)\eta(z_k)) \cup \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}^*_S(\eta(a))$$

$$\subseteq \widetilde{\delta}_S(\eta(a)) \cup \widetilde{\delta}^*_S(\eta(a))$$

$$= \left(\widetilde{\delta}_S \cup \widetilde{\delta}^*_S\right)(\eta(a))$$

Hence $\widetilde{\gamma}_R \diamond \widetilde{\gamma}^*_R \sqsupseteq \widetilde{\gamma}_R \widetilde{\cap} \widetilde{\gamma}^*_R$ and $\widetilde{\delta}_S \diamond \widetilde{\delta}^*_S \sqsupseteq \widetilde{\delta}_S \widetilde{\cup} \widetilde{\delta}^*_S$.

Also from Lemma 6.4 $\widetilde{\gamma}_R \diamond \widetilde{\gamma}^*_R \sqsubseteq \widetilde{\gamma}_R \widetilde{\cap} \widetilde{\gamma}^*_R$ and $\widetilde{\delta}_S \diamond \widetilde{\delta}^*_S \sqsubseteq \widetilde{\delta}_S \widetilde{\cup} \widetilde{\delta}^*_S$. Therefore, $\widetilde{\gamma}_R \diamond \widetilde{\gamma}^*_R = \widetilde{\gamma}_R \widetilde{\cap} \widetilde{\gamma}^*_R$ and $\widetilde{\delta}_S \diamond \widetilde{\delta}^*_S = \widetilde{\delta}_S \widetilde{\cup} \widetilde{\delta}^*_S$.

(ii)⟹(i) Let L_1 and L_2 be right h-ideal and left h-ideals of R respectively and M_1 and M_2 be right h-ideal and left h-ideal of S respectively. Then by the Proposition 5.5, $\widetilde{\chi}_{L_1 \rightrightarrows M_1}$ is right BSI-h-ideal of $E = R \cup S$ and $\widetilde{\chi}_{L_2 \rightrightarrows M_2}$ is left BSI-h-ideal of $E = R \cup S$ over U. By Lemma 6.2, we obtain $L_1 \cap L_2 \supseteq \overline{L_1 L_2}$ and $M_1 \cap M_2 \supseteq \overline{M_1 M_2}$. Let $a \in L_1 \cap L_2$ and $\eta(a) \in M_1 \cap M_2$, then $\widetilde{\omega}_{L_1}(a) = \widetilde{\omega}_{L_2}(a) = \widetilde{U}_R$, $\widetilde{\varkappa}_{M_1}(a) = \widetilde{\varkappa}_{M_2}(a) = \widetilde{\emptyset}_S$ and so $(\widetilde{\omega}_{L_1} \diamond \widetilde{\omega}_{L_2})(a) = \left(\widetilde{\omega}_{L_1} \widetilde{\cap} \widetilde{\omega}_{L_2}\right)(a) = \widetilde{U}_R$, $\left(\widetilde{\delta}_{M_1} \diamond \widetilde{\delta}_{M_2}\right)(a) =$

$\left(\widetilde{\delta}_{M_1}\widetilde{\cup}\widetilde{\delta}^*_{M_2}\right)(a) = \widetilde{\emptyset}_S$. Now, for a, x_1 ,x_2 , $b \in R$ and $\eta(a), \eta(x_1)$,$\eta(x_2)$, $\eta(b) \in S$ we have $a + x_1x_2 + b = y_1y_2 + b$ and $\eta(a) + \eta(x_1)\eta(x_2) + \eta(b) = \eta(y_1)\eta(y_2) + \eta(b)$. This implies, $\widetilde{\omega}_{L_1}(x_1) = \widetilde{\omega}_{L_1}(x_2) = \widetilde{\omega}_{L_2}(y_1) = \widetilde{\omega}_{L_2}(y_2) = \widetilde{U}_R$, $\widetilde{\varkappa}_{M_1}(\eta(x_1)) = \widetilde{\varkappa}_{M_1}(\eta(x_2)) = \widetilde{\varkappa}_{M_2}(\eta(y_1)) = \widetilde{\varkappa}_{M_2}(\eta(y_2)) = \widetilde{\emptyset}_S$.Thus, $x_1, x_2 \in L_1$, $y_1, y_2 \in L_2$ and $\eta(x_1), \eta(x_2) \in M_1, \eta(y_1), \eta(y_2) \in M_2$. Therefore, $L_1 \cap L_2 \subseteq \overline{L_1L_2}$, $M_1 \cap M_2 \subseteq \overline{M_1M_2}$,and so $L_1 \cap L_2 = \overline{L_1L_2}$, $M_1 \cap M_2 = \overline{M_1M_2}$. By Lemma 6.3, both R nad S are h-hemi-regular. Hence condition *(i)* holds. □

7 Conclusions

In this paper, we investigated BSI-h-hemi-rings and BSI-h-ideals in the union of two isomorphic hemi-rings. The notion of BS-set will evolve a foundation to handle the problems in other algebraic structures and especially in two different algebraic structures. Therefore, this paper gives an idea for the beginning of a new study for approximations of data with uncertainties. This concept provides a new soft algebraic tool in many uncertainties problems. In future, we will apply BS-set (i) in probability via BS-falling ideals of hemi-rings (ii) in soft topological vector spaces (iii) in decision making problems (iv) in different algebraic structures.

Competing Interests

The authors declare that no competing interests exist.

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