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Moments of generalized order statistics for Pareto-Rayleigh distribution

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Abstract: In this paper, we derive the explicit expressions for single and product moments of generalized order statistics from Pareto-Rayleigh distribution using hypergeometric functions. Also, some interesting remarks are presented.

Keywords: Generalized order statistics; Pareto-Rayleigh distribution; Single moments; Product moments; Hypergeometric functions.

MSC: 62G30; 33C90.

1. Introduction

Kamps [1] introduced the concept of generalized order statistics (gos) as follows: Let us note $n \in N$, $k \geq 1$, and $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \Re^{n-1}$, $1 \leq r \leq n-1$, such that

$$\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j > 0 \quad \text{for} \quad 1 \leq r \leq n-1.$$

The random variables $X(1, n, \tilde{m}, k), X(2, n, \tilde{m}, k), \dots, X(n, n, \tilde{m}, k)$ are said to be gos from a continuous population with cumulative distribution function (cdf) $F(x)$ and probability distribution function (pdf) $f(x)$ if their joint pdf is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}(x_i)\right]^{m_i} f(x_i)\right)\left[\bar{F}(x_n)\right]^{k-1} f(x_n), \tag{1}$$

defined on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$ of $\Re^n$, where $\bar{F}(x) = 1 - F(x)$.

The model of gos contains special cases such as ordinary order statistics $(\gamma_i = n - i + 1; i = 1, 2, \dots, n$ *i.e.* $m_1 = m_2 = \cdots = m_{n-1} = 0, k = 1)$, k-th record values $(\gamma_i = k$ *i.e.*, $m_1 = m_2 = \cdots = m_{n-1} = -1, k \in N)$, sequential order statistics $(\gamma_i = (n - i + 1)\alpha_i; \alpha_1, \alpha_2, \dots, \alpha_n > 0)$, order statistics with non-integer sample size $(\gamma_i = \alpha - i + 1; \alpha > 0)$, Pfeifer's record values $(\gamma_i = \beta_i; \beta_1, \beta_2, \dots, \beta_n > 0)$ and progressive type II censored order statistics $(\gamma_r = n - r + 1 + \sum_{i=r}^{l} m_i, 1 \leq r \leq l \leq n, m_i \in N, k = m_n + 1)$, see [1–3].

Here we shall obtain the results for $\gamma_i \neq \gamma_j$ and then deduce the results for $\gamma_i = \gamma_j$ $(m_1 = m_2 = \cdots = m_{n-1} = m \neq -1)$.

Therefore, we will consider two cases:
Case I: $\gamma_i = \gamma_j$ $(m_1 = m_2 = \cdots = m_{n-1} = m \neq -1)$ [1].
Case II: $\gamma_i \neq \gamma_j$, $i \neq j$ $i, j = 1, 2, \dots, n-1$ [2].
Case I: The pdf of $r-$th gos $X(r, n, m, k)$, is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)), \tag{2}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is given by

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}\,[\bar{F}(x)]^m\, g_m^{r-1}(F(x)) \tag{3}$$

$$\times\,[h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[\bar{F}(y)]^{\gamma_s-1}\,f(x)\,f(y),\;\; x<y, \tag{4}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & ,\;\; m \ne -1 \\ -\ln(1-x) & ,\;\; m = -1 \end{cases}$$

and

$$g_m(x) \;=\; h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \;\; x \in [0,1).$$

Case II: The pdf of $r-$th gos $X(r,n,\tilde{m},k)$, is given by

$$f_{X(r,n,\tilde{m},k)}(x) \;=\; C_{r-1}\sum_{i=1}^{r} a_i(r)\,[\bar{F}(x)]^{\gamma_i-1}\,f(x) \tag{5}$$

with the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$,

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) \;=\; C_{s-1}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}^{\gamma_i}\right]\left[\sum_{i=1}^{r} a_i(r)\{\bar{F}(x)\}^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\,\frac{f(y)}{\bar{F}(y)} \tag{6}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i,$$

$$\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j,$$

$$a_i(r) \;=\; \prod_{j=1}^{r}\frac{1}{(\gamma_j-\gamma_i)}, \quad j \ne i, \quad \gamma_j \ne \gamma_i, \quad 1 \le i \le r \le n,$$

$$a_i^{(r)}(s) \;=\; \prod_{j=r+1}^{n}\frac{1}{(\gamma_j-\gamma_i)}, \quad j \ne i, \quad \gamma_j \ne \gamma_i, \quad r+1 \le i \le s \le n.$$

For $m_1 = m_2 = \cdots = m_{n-1} = m \ne -1$, it can be shown that [3]:

$$a_i(r) = \frac{(-1)^{r-i}}{(m+1)^{r-1}\,(r-1)!}\binom{r-1}{r-i} \tag{7}$$

and

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}\,(s-r-1)!}\binom{s-r-1}{s-i}. \tag{8}$$

In this paper we are interested in a situation when a random variable X follows the Pareto-Rayleigh(P-R) distribution with pdf

$$f(x;\alpha,\sigma) = \frac{\alpha}{\sigma^2}\,x\left(1+\frac{x^2}{2\sigma^2}\right)^{-(\alpha+1)} \quad x>0,\ \alpha>1, \text{ and } \sigma>0, \tag{9}$$

and with df

$$F(x;\alpha,\sigma) = 1 - \left(1+\frac{x^2}{2\sigma^2}\right)^{-\alpha} \quad x>0,\ \alpha>1, \text{ and } \sigma>0. \tag{10}$$

In view of (8) and (9),

$$(1+\frac{x^2}{2\sigma^2})\,f(x) = \frac{\alpha}{\sigma^2}\,x\,\bar{F}(x) \tag{11}$$

Pareto-Rayleigh distribution can be seen as a member of Transformed-Transformer family (or T-X family) of distributions proposed by Alzaatreh *et al.*, [4]. This distribution is recognized as a good model for fitting various lifetime data, see Jebeli and Deiri [5]. This is also confirmed in [6] were a comparative study on the performance of Pareto-Rayleigh distribution against biased Lomax distribution was conducted. Further, for more details on Pareto-Rayleigh distribution one can see [7–9].

Exact moments expressions of gos for different distributions have been obtained by literature. Some examples are exponentiated Log-logistic distribution, Burr type XII distribution, linear exponential distribution, Erlang-truncated exponential distribution, Burr distribution, power function distribution, type II exponentiated Log-logistic distribution, extended exponential distribution, generalized Pareto distribution, q-Weibull distribution; see, respectively, Athar and Nayabuddin [10], Khan *et al.*, [11], Ahmad [12], Khan *et al.*, [13], Khan and Khan [3], Kumar and Khan [14], Kumar [15], Kumar and Dey [16], Malik and Kumar [17], Singh *et al.*, [18] and Kumar *et al.*, [19].

In this paper, we have derived explicit expression for single and product moments of Pareto-Rayleigh distribution based on gos.

2. Relations for Product Moments

In this section, we derive the exact expressions for product moments of generalized order statistics in the following theorems. Before coming to the main result, the following lemma is proved.

Lemma 1. *For the Pareto-Rayleigh distribution with cdf (1.9) next relations holds*

$$\Phi_{j,l}(a,b) = \frac{(2\sigma^2)^{\left(\frac{j+l}{2}+2\right)}}{2\,(j+2)}\, B\Big(\frac{j+l}{2}+2,\, \alpha\, b - \frac{l}{2}\Big)\, {}_3F_2\Big(\frac{j}{2}+1,\, 1-a\,\alpha+\frac{j}{2},\, \frac{j+l}{2}+2;\, \frac{j}{2}+2,\, \frac{j}{2}+\alpha\, b+2; 1\Big) \quad (12)$$

where

$$\Phi_{j,l}(a,b) = \int_0^\infty \int_0^y \frac{x^{j+1}}{(1+\frac{x^2}{2\sigma^2})^{a\,\alpha+1}}\, \frac{y^{l+1}}{(1+\frac{y^2}{2\sigma^2})^{\alpha\, b+1}} dx\, dy$$

and

$${}_pF_q[a_1,\dots,a_p; b_1,\dots,b_q; x] = \sum_{r=0}^{\infty}\Big[\prod_{j=1}^{p}\frac{\Gamma(a_j+r)}{\Gamma(a_j)}\Big]\Big[\prod_{j=1}^{q}\frac{\Gamma(b_j)}{\Gamma(b_j+r)}\Big]\frac{x^r}{r!},$$

for $p=q+1$ and $\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j > 0$.

Proof. We have

$$\Phi_{j,l}(a,b) = \int_0^\infty \frac{y^{l+1}}{(1+\frac{y^2}{2\sigma^2})^{\alpha\, b+1}} \left[\int_0^y \frac{x^{j+1}}{(1+\frac{x^2}{2\sigma^2})^{a\,\alpha+1}} dx\right] dy \quad (13)$$

Let

$$B(y) = \int_0^y \frac{x^{j+1}}{(1+\frac{x^2}{2\sigma^2})^{a\,\alpha+1}} dx \quad (14)$$

Substituting $1-u = \frac{1}{(1+\frac{x^2}{2\sigma^2})}$ in (14), we get

$$\begin{aligned} B(y) &= \frac{(2\sigma^2)^{\left(1+\frac{j}{2}\right)}}{2} \int_0^{\frac{\frac{y^2}{2\sigma^2}}{\left(1+\frac{y^2}{2\sigma^2}\right)}} u^{\frac{j}{2}}\,(1-u)^{a\,\alpha-\frac{j}{2}-1}\, du \\ &= \frac{(2\sigma^2)^{\left(1+\frac{j}{2}\right)}}{2}\, B_{\frac{\frac{y^2}{2\sigma^2}}{\left(1+\frac{y^2}{2\sigma^2}\right)}}\Big(\frac{j}{2}+1,\, a\alpha-\frac{j}{2}\Big). \end{aligned}$$

From (13), we have

$$\Phi_{j,l}(a,b) = \frac{(2\sigma^2)^{\left(1+\frac{j}{2}\right)}}{2} \int_0^\infty \frac{y^{l+1}}{(1+\frac{y^2}{2\sigma^2})^{\alpha\, b+1}}\, B_{\frac{\frac{y^2}{2\sigma^2}}{\left(1+\frac{y^2}{2\sigma^2}\right)}}\Big(\frac{j}{2}+1,\, a\alpha-\frac{j}{2}\Big)\, dy, \quad (15)$$

where $B_x(p,q) = \int_0^x u^{p-1}\,(1-u)^{q-1}\,du$. We know that

$$B_x(p,q) = p^{-1}\,x^p\,{}_2F_1\,(p, 1-q;\; p+1;\; x) \tag{16}$$

and

$$\int_0^1 u^{a-1}\,(1-u)^{b-1}\,{}_2F_1\,(c,d;\; e;\; u)\,du = B(a,b)\,{}_3F_2\,(c,d,a;\; e, a+b;\; 1) \tag{17}$$

Substituting (16) and (17) in (15), we get

$$\begin{aligned}\Phi_{j,l}(a,b) =&\frac{(2\sigma^2)^{\left(1+\frac{j}{2}\right)}}{2}\int_0^\infty \frac{y^{l+1}}{(1+\frac{y^2}{2\sigma^2})^{\alpha\, b+1}}\left(\frac{\frac{y^2}{2\sigma^2}}{1+\frac{y^2}{2\sigma^2}}\right)^{\frac{j}{2}+1}\left(\frac{j}{2}+1\right)^{-1}\\ &{}_2F_1\left[\frac{j}{2}+1,\, 1-a\,\alpha+\frac{j}{2},;\; \frac{j}{2}+2;\; \left(\frac{\frac{y^2}{2\sigma^2}}{1+\frac{y^2}{2\sigma^2}}\right)\right]\,dy.\end{aligned} \tag{18}$$

Setting $t = \frac{\frac{y^2}{2\sigma^2}}{1+\frac{y^2}{2\sigma^2}}$ in (18), we get

$$\begin{aligned}\Phi_{j,l}(a,b) =&\frac{(2\sigma^2)^{\left(\frac{j+l}{2}+2\right)}}{2\,(j+2)}\int_0^1 t^{\frac{j+l}{2}+1}\,(1-t)^{\alpha\, b-\frac{l}{2}-1}\,{}_2F_1\left[\frac{j}{2}+1,\, 1-a\,\alpha+\frac{j}{2},;\; \frac{j}{2}+2;\; t\right]\,dt\\ =&\frac{(2\sigma^2)^{\left(\frac{j+l}{2}+2\right)}}{2\,(j+2)}\,B\left(\frac{j+l}{2}+2,\, \alpha\, b-\frac{l}{2}\right){}_3F_2\left(\frac{j}{2}+1,\, 1-a\,\alpha+\frac{j}{2},\, \frac{j+l}{2}+2;\; \frac{j}{2}+2,\, \frac{j}{2}+\alpha\, b+2; 1\right).\end{aligned}$$

□

Lemma 2. *Setting $j=0$ or $l=0$ in Lemma 1, we obtain*

$$\Phi_{0,l}(a,b) = \frac{\sigma^2}{a\,\alpha}\,[\Phi_l(b) - \Phi_l(a+b)] \tag{19}$$

and

$$\Phi_{j,0}(a,b) = \frac{\sigma^2}{b\,\alpha}\,[\Phi_j(a+b)] \tag{20}$$

where

$$\Phi_j(a) = \int_0^\infty \frac{x^{j+1}}{\left(1+\frac{x^2}{2\sigma}\right)^{a\,\alpha+1}}\quad dx = \frac{(2\sigma^2)^{\left(1+\frac{j}{2}\right)}}{2}\,B\left(a\alpha-\frac{j}{2}, 1+\frac{j}{2}\right).$$

Proof. Substituting $j=0$ in (13), we get

$$\begin{aligned}\Phi_{0,l}(a,b) &= \int_0^\infty \frac{y^{l+1}}{\left(1+\frac{y^2}{2\sigma^2}\right)^{\alpha\, b+1}}\left[\int_0^y \frac{x}{\left(1+\frac{x^2}{2\sigma^2}\right)^{a\,\alpha+1}}dx\right]\,dy\\ &= \frac{\sigma^2}{a\,\alpha}\int_0^\infty \frac{y^{l+1}}{\left(1+\frac{y^2}{2\sigma^2}\right)^{\alpha\, b+1}}\left[1-\frac{1}{\left(1+\frac{y^2}{2\sigma^2}\right)^{a\,\alpha}}\right]\,dy\\ &= \frac{\sigma^2}{a\,\alpha}\,[\Phi_j(b) - \Phi_l(a+b)].\end{aligned}$$

Similarly, we get (20) by noting that

$${}_3F_2(a,\, b,\, c;\, c,\, d; 1) ={}_2 F_1(a,\, b;\, d; 1) = \frac{\Gamma(d)\,\Gamma(d-a-b)}{\Gamma(d-a)\,\Gamma(d-b)}.$$

□

Theorem 1. *Generalized product moments for Pareto-Rayleigh distribution are given as*

$$\mu_{r,s,n,\tilde{m},k}^{(j,l)} = E[X^j(r,n,\tilde{m},k)\,X^l(s,n,\tilde{m},k)] = C_{s-1}\left(\frac{\alpha}{\sigma^2}\right)^2\left[\sum_{t=r+1}^{s} a_t^{(r)}(s)\left(\sum_{i=1}^{r} a_i(r)\,\Phi_{j,l}(\gamma_i-\gamma_t,\,\gamma_t)\right)\right]. \tag{21}$$

Proof. We have

$$\mu_{r,s,n,\tilde{m},k}^{(j,l)} = C_{s-1}\int_0^\infty\int_0^y x^j y^l\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}^{\gamma_i}\right]\left(\sum_{i=1}^{r} a_i(r)\,\{\bar{F}(x)\}^{\gamma_i}\right)\frac{f(x)}{\bar{F}(x)}\,\frac{f(y)}{\bar{F}(y)}\,dx\,dy.$$

which yields (21). □

Corollary 2. *Product moment for Pareto-Rayleigh distribution, when $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$ is given as*

$$\begin{aligned}\mu_{r,s,n,m,k}^{(j,l)} =& E\left[X^j(r,n,m,k)\,X^l(s,n,m,k)\right]\\ =& \frac{C_{s-1}}{(r-1)!\,(s-r-1)!\,(m+1)^{s-2}}\left(\frac{\alpha}{\sigma^2}\right)^2\sum_{i=0}^{r-1}\sum_{t=0}^{s-r-1}(-1)^{i+t}\binom{r-1}{i}\binom{s-r-1}{t}\Phi_{j,l}(\gamma_{r-i}-\gamma_{s-t},\,\gamma_{s-t}).\end{aligned} \tag{22}$$

Remark 1. Setting $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = 1$ in (22), we get the result as the product moment of order statistics as

$$\begin{aligned}\mu_{r,s,n,0,1}^{(j,l)} &= \mu_{r,s:n}^{j,l}\\ &= \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}\left(\frac{\alpha}{\sigma^2}\right)^2\sum_{i=0}^{r-1}\sum_{t=0}^{s-r-1}(-1)^{i+t}\binom{r-1}{i}\binom{s-r-1}{t}\Phi_{j,l}(s-r-t+i,\,n-s+t+1).\end{aligned} \tag{23}$$

Corollary 3. *Single moments of the Pareto-Rayleigh distribution are of the form*

$$\mu_{s,n,\tilde{m},k}^{(l)} = C_{s-1}\left(\frac{\alpha}{\sigma^2}\right)\sum_{i=1}^{s} a_i(s)\,\Phi_l(\gamma_i). \tag{24}$$

Proof. Putting $j = 0$ in (21) and using (19), we get

$$\begin{aligned}\mu_{r,s,n,\tilde{m},k}^{(l)} =& C_{s-1}\left(\frac{\alpha}{\sigma^2}\right)\left[\sum_{t=r+1}^{s}\frac{a_t^{(r)}(s)}{(\gamma_i-\gamma_t)}\left(\sum_{i=1}^{r} a_i(r)\,\{\Phi_l(\gamma_t)-\Phi_l(\gamma_i)\}\right)\right].\\ \mu_{s,n,\tilde{m},k}^{(l)} =& C_{s-1}\left(\frac{\alpha}{\sigma^2}\right)\left[\sum_{t=r+1}^{s} a_t^{(r)}(s)\,\Phi_l(\gamma_t)\left(\sum_{i=1}^{r}\frac{a_i(r)}{(\gamma_i-\gamma_t)}\right)\right]\\ &+ C_{s-1}\left(\frac{\alpha}{\sigma^2}\right)\left[\sum_{i=1}^{r} a_i(r)\,\Phi_l(\gamma_i)\left(\sum_{t=r+1}^{s}\frac{a_t^{(r)}(s)}{(\gamma_i-\gamma_t)}\right)\right].\end{aligned}$$

Now using the results found in [20] we obtain

$$\sum_{i=1}^{r}\frac{a_i(r)}{(\gamma_i-\gamma_j)} = \prod_{j=1}^{r}\frac{1}{(\gamma_i-\gamma_j)},\quad j\neq i,\quad \gamma_j\neq\gamma_i,\quad 1\le i\le r\le n,$$

and

$$\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{(\gamma_i-\gamma_j)} = \prod_{j=r+1}^{s}\frac{1}{(\gamma_i-\gamma_j)},\quad j\neq i,\quad \gamma_j\neq\gamma_i,\quad r+1\le i\le s\le n.$$

Hence,

$$\mu_{s,n,\tilde{m},k}^{(l)} = C_{s-1}\left(\frac{\alpha}{\sigma^2}\right)\left[\sum_{t=r+1}^{s} a_t^{(r)}(s)\,\Phi_l(\gamma_t)\left(\prod_{j=1}^{r}\frac{1}{(\gamma_i-\gamma_j)}\right)\right]$$

$$+ C_{s-1}\left(\frac{\alpha}{\sigma^2}\right)\left[\sum_{i=1}^{r} a_i(r)\,\Phi_l(\gamma_i)\left(\prod_{j=r+1}^{s}\frac{1}{(\gamma_i-\gamma_j)}\right)\right],$$

which yields (24). □

Corollary 4. *Corollary2.3 Single moments of gos for Pareto-Rayleigh distribution, when $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$, are given as*

$$\mu^{(l)}_{s,n,m,k} = \frac{C_{s-1}}{(s-1)!}\,\frac{1}{(m+1)^{s-1}}\left(\frac{\alpha}{\sigma^2}\right)\sum_{i=0}^{s-1}(-1)^i\binom{r-1}{i}\Phi_j(\gamma_{s-i}). \tag{25}$$

Proof. Setting $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$ in (24) and using (7) we get the result as the single moment. □

Remark 2. Setting $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = 1$ in (25), we get the result as the single moment from order statistics

$$\mu^{(l)}_{s,n,0,1} = \mu^{(l)}_{s:n} = \frac{C_{s-1}}{(s-1)!}\left(\frac{\alpha}{\sigma^2}\right)\sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}\Phi_j(n-s+i+1). \tag{26}$$

Remark 3. Setting $j = 0$ and $l = 0$ in (21) we get

$$\sum_{i=1}^{r}\sum_{t=r+1}^{s}\frac{a_i(r)\,a^r_t(s)}{\gamma_i\,\gamma_t} = \frac{1}{C_{s-1}}, \tag{27}$$

and setting $l = 0$ in (24) we obtain

$$\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i} = \frac{1}{C_{r-1}}. \tag{28}$$

Combining (27) and (28), we get another identity,

$$\sum_{t=r+1}^{s}\frac{a^r_t(s)}{\gamma_t} = \frac{C_{r-1}}{C_{s-1}}. \tag{29}$$

When $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$, (29) reduces to another identity

$$\sum_{t=0}^{s-r-1}(-1)^t\binom{s-r-1}{t}\frac{1}{\gamma_{s-t}} = \frac{C_{r-1}(s-r-1)\,!\,(m+1)^{s-r-1}}{C_{s-1}}, \tag{30}$$

which is obtained in [3].

Remark 4. Setting $\gamma_r = k+n-r+\sum_{i=r}^{l} m_j$, $1 \leq r \leq l \leq n$, $m_i \in N$, in (21), then the product moments of progressive type II censored order statistics of Pareto-Rayleigh distribution can be obtained.

3. Numerical Computations

Here we have calculated means and variances for order statistics (Table 1 & 2), and generalized order statistics (gos) (Table 3 & 4). All computations here we obtained using Mathematica. Mathematica like other algebraic manipulation packages allow for arbitrary precisions, so the accuracy of the given values is not an issue. In case of order statistics, the relation $\sum_{r=1}^{n}\mu^{j}_{r,n,0,1} = n\,\mu^{j}_{1,1,0,1}$, $\quad j = 1, 2$, is used to evaluate the means and variancess, see [21]. It is observed that when the sample size n is fixes, increasing the value of r directly increases the means and variances, whereas, for fixed r, the opposite occurs in the case when the sample size n increases.

Table 1. Means of order statistics from Pareto-Rayleigh distribution (α=2, σ=1)

	n							
r	1	2	3	4	5	6	7	8
1	1.1107	0.6942	0.5469	0.4653	0.4120	0.3736	0.3443	0.3209
2		1.5272	0.9892	0.7907	0.6786	0.6040	0.5496	0.5078
3			1.7962	1.1877	0.9589	0.8279	0.7398	0.6752
4				1.9991	1.3403	1.0900	0.9453	0.8474
5					2.1638	1.4654	1.1984	1.0433
6						2.3035	1.5722	1.2916
7							2.4253	1.6658
8								2.5339

Table 2. Variances of order statistics from Pareto-Rayleigh distribution (α=2, σ=1)

	n							
r	1	2	3	4	5	6	7	8
1	0.7663	0.1847	0.1011	0.0692	0.0525	0.0422	0.0353	0.0303
2		1.0009	0.2214	0.1176	0.0792	0.0594	0.0475	0.0395
3			1.1735	0.2464	0.1281	0.0852	0.0635	0.0504
4				1.3180	0.2672	0.1366	0.0900	0.0666
5					1.4450	0.2855	0.1441	0.0942
6						1.5599	0.3021	0.1509
7							1.6655	0.3175
8								1.7639

Table 3. Means of gos from Pareto-Rayleigh distribution (α=2, σ=1, m=1, k=2)

	n							
r	1	2	3	4	5	6	7	8
1	0.3471	0.2327	0.1868	0.1605	0.1428	0.1300	0.1200	0.1121
2		0.2308	0.1622	0.1329	0.1155	0.1036	0.0947	0.0878
3			0.1325	0.0957	0.0795	0.0697	0.0628	0.0577
4				0.0724	0.0532	0.0447	0.0394	0.0357
5					0.0386	0.0288	0.0243	0.0216
6						0.0203	0.0153	0.0130
7							0.0105	0.0080
8								0.0054

Table 4. Variances of gos from Pareto-Rayleigh distribution (α=2, σ=1, m=1, k=2)

	n							
r	1	2	3	4	5	6	7	8
1	0.2128	0.0887	0.0560	0.0409	0.0322	0.0266	0.0226	0.0197
2		0.2086	0.0971	0.0641	0.0481	0.0385	0.0321	0.0275
3			0.1480	0.0733	0.0499	0.0380	0.0308	0.0259
4				0.0914	0.0471	0.0327	0.0253	0.0207
5					0.0527	0.0280	0.0197	0.0154
6						0.0292	0.0158	0.0113
7							0.0158	0.0087
8								0.0084

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