# An Approach to Multiple Attribute Group Decision Making under Hesitant Fuzzy Linguistic Environments with Incomplete Weight Information

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***Author's contribution***

*Author ZZ designed the study, performed the statistical analysis, wrote the protocol, and wrote the overall draft of the manuscript. The author read and approved the final manuscript.*

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## ABSTRACT

As a combination of the fuzzy linguistic approach and the hesitant fuzzy set (HFS), hesitant fuzzy linguistic term set (HFLTS) is an efficient tool to deal with situations in which experts hesitate between several possible linguistic terms to assess the membership of an element in qualitative settings. In this paper, we develop a novel method for solving multiple attribute group decision making (MAGDM) problems with hesitant fuzzy linguistic information, in which the attribute values provided by the decision makers take the form of hesitant fuzzy linguistic term sets (HFLTSs), the information about the weights of decision makers is unknown, and the information about attribute weights is incompletely known or completely unknown. The developed method consists of three parts. The first one establishes a quadratic programming model based on the maximizing group consensus method, which can be used to determine the weights of decision makers. The second one uses the maximizing deviation method to establish an optimization model, from which the optimal weights of attributes can be derived. The third one extends the TOPSIS method to hesitant fuzzy linguistic environments and develops a hesitant fuzzy linguistic TOPSIS method, which determines a solution with the shortest distance to the hesitant fuzzy linguistic positive ideal solution (PIS) and the greatest distance from the hesitant fuzzy linguistic negative ideal solution (NIS). Moreover, a

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practical example is provided to illustrate the proposed method. Finally, the comparison analysis with the other methods shows that the developed method is very effective and appropriate for solving hesitant fuzzy linguistic MAGDM with incomplete weight information.

*Keywords: Hesitant fuzzy set; hesitant fuzzy linguistic term set; multiple attribute group decision making; maximizing group consensus method; maximizing deviation method; TOPSIS.*

## 1. INTRODUCTION

Owing to the fact that the difficulty of establishing the membership degree of an element to a set is sometimes not because we have a margin of error (as in intuitionistic fuzzy set [1], interval-valued fuzzy set [2], or interval-valued intuitionistic fuzzy set [3]) or some possibility distribution on the possible values (as in type-2 fuzzy set [4]), but because we have some possible numerical values [5], Torra [5] presented a new concept of hesitant fuzzy set (HFS), in which several numerical values between 0 and 1 are simultaneously used to represent the membership degree of an element to a given set. As a result, hesitant fuzzy set is not only an extension of fuzzy sets [6] to deal with uncertainty but also an efficient tool that can represent situations in which several membership functions for a fuzzy set are possible. Since its appearance, hesitant fuzzy set have attracted more and more attentions [7-24]. Especially, hesitant fuzzy set theory has been successfully applied to multiple attribute group decision making (MAGDM) in which the attribute values take the form of hesitant fuzzy elements (HFEs) [15] that are expressed as a set of several possible numerical values.

However, it is noted that the attribute values about alternatives are usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human thinking [25,26]; thus, a qualitative form may be more appropriate for assessing the information than a quantitative form. For example, some linguistic terms like "very good", "good", "slightly good", and "fair" rather than some numerical values are frequently used to evaluate the "comfort" or "design" of a car [27]. The fuzzy linguistic approach is a very efficient approximate technique to deal with such cases [28,29]. It is noticed that the fuzzy linguistic approach usually uses a single term to express the information regarding a linguistic variable [30,31]. But in practical applications, the decision makers (DMs) may think of several possible linguistic values at the same time or richer expressions rather than a single term for an indicator, alternative, variable, etc [30,31]. For example, when evaluating the "comfort" or "design" of a car, some linguistic expressions such as "between very good and slightly good", "greater than fair", and "lower than good" are often used. For the sake of a better description of this situation, Rodríguez et al. [30] introduced the concept of a hesitant fuzzy linguistic term set (HFLTS), which can efficiently deal with the situations in which the DMs hesitate between several linguistic terms to assess an indicator, alternative, variable, etc. Rodríguez et al. [30] presented a multi-criteria linguistic decision making model based on hesitant fuzzy linguistic term sets (HFLTSs). Furthermore, Rodríguez et al. [31] proposed a group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. Zhang and Wu [32] defined several hesitant fuzzy linguistic aggregation operators for aggregating the input arguments that take the form of HFLTSs, and then utilized these operators to develop an approach for multiple attribute group decision making with hesitant fuzzy linguistic information. Obviously, these researches [30,31,32] put their emphasis on the aggregation techniques in MAGDM under hesitant fuzzy linguistic situations, which have

some limitations as follows: (1) when using these techniques, the weight vectors of decision makers and attributes are proposed by the DMs in advance, and thus are more or less subjective and insufficient; (2) when using these techniques, the dimension of the aggregated HFLTSs may increase, which may increase the computational complexity and lead to the loss of decision information.

In fact, in many MAGDM with hesitant fuzzy linguistic information, because of time pressure, lack of knowledge or data, and the decision makers' limited expertise about the problem domain [33], the information about the weights of decision makers is unknown, and the information about the attribute weights is incompletely known or completely unknown. Considering that the aforementioned works [30,31,32] are inappropriate for dealing with such situations, in this paper, we establish a quadratic programming model based on the maximizing group consensus method to objectively determine the weights of decision makers. We further use the maximizing deviation method to establish an optimization model, based on which the optimal attribute weights can be objectively obtained. Moreover, motivated by the TOPSIS, we develop an extended TOPSIS method to determine the optimal alternative, which includes two stages. The first stage is called the hesitant fuzzy linguistic TOPSIS, which can be used to calculate the individual relative closeness coefficient of each alternative to the individual hesitant fuzzy linguistic positive ideal solution (PIS). The second stage is the standard TOPSIS, which is used to calculate the group relative-closeness coefficient of each alternative to group PIS and select the optimal one with the maximum group relative-closeness coefficient.

To do so, the remainder of this paper is organized as follows. In Section 2, we briefly review some concepts related to the fuzzy linguistic approach, HFSs and HFLTSs. Section 3 develops a novel method based on the maximizing group consensus method, the maximizing deviation method and TOPSIS for solving the hesitant fuzzy linguistic MAGDM problem with incomplete weight information. An illustrative example is provided to show the effectiveness and practicality of the developed method in Section 4. In the sequel, Section 5 makes a comparison analysis with the other method. This paper ends with some concluding remarks in Section 6.

## 2. PRELIMINARIES

### 2.1 The Fuzzy Linguistic Approach

The fuzzy linguistic approach is an approximate technique, which represents qualitative aspects as linguistic values by means of linguistic variables. Let $S = \left\{ s_i \middle| i = 0, 1, 2, \cdots, g \right\}$ be a finite and totally ordered discrete linguistic term set with odd cardinality, where $s_i$ represents a possible value for a linguistic variable, $g$ is the number of granularity in the linguistic term set, which is a positive integer. For example, a set of nine terms $S$ could be given as follows [34-38]:

$$S = \begin{cases} s_0 : \text{extremely poor}, \; s_1 : \text{very poor}, \; s_2 : \text{poor}, \; s_3 : \text{slightly poor}, \; s_4 : \text{fair,} \\ s_5 : \text{slightly good}, \; s_6 : \text{good}, \; s_7 : \text{very good}, \; s_8 : \text{extremely good} \end{cases}$$

Usually, it is required that linguistic term set $S$ should satisfy the following characteristics:

(1) The set is ordered: $s_i \geq s_j$ if $i \geq j$ ;

(2) There is the negation operator: $\text{neg}(s_i) = s_j$ such that $j = g - i$ ( $g + 1$ is the granularity of the term set);

(3) Max operator: $\max(s_i, s_j) = s_i$ if $s_i \geq s_j$ ;

(4) Min operator: $\min(s_i, s_j) = s_i$ if $s_i \leq s_j$ .

To preserve all the given information, Xu [39,40] extended the discrete linguistic term set $S$ to a continuous linguistic term set $\overline{S} = \{s_\alpha \mid s_0 \leq s_\alpha \leq s_g, \alpha \in [0, g]\}$ . If $s_\alpha \in S$ , then $s_\alpha$ is called an original linguistic term, otherwise, $s_\alpha$ is called a virtual linguistic term. In general, the decision maker uses the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in operation.

Considering any two linguistic terms $s_\alpha, s_\beta \in \overline{S}$ , and $\lambda \in [0,1]$ , Xu [39,40] defined two operational laws as follows:

(1) $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha = s_{\alpha+\beta}$ ;

(2) $\lambda s_\alpha = s_{\lambda\alpha}$ .

We denote $I(s)$ as the position index of $s$ in $\overline{S}$ : For example, $I(s_\alpha) = \alpha$ .

Let $s_\alpha, s_\beta \in S$ be two extended linguistic terms, then Xu [41] defined the distance between $s_\alpha$ and $s_\beta$ as follows:

$$d(s_\alpha, s_\beta) = \frac{|I(s_\alpha) - I(s_\beta)|}{g} = \frac{|\alpha - \beta|}{g}$$

(1)

## 2.2 Hesitant Fuzzy Sets (HFSs)

Torra [5] proposed the notion of hesitant fuzzy sets to manage the situations in which several numerical values are possible for the definition of the membership of an element to a given set.

**Definition 2.1 [5].** Let $X$ be a reference set, a hesitant fuzzy set (HFS) on $X$ is in terms of a function $h$ that when applied to $X$ returns a subset of $[0,1]$ .

To be easily understood, Xia and Xu [15] expressed the HFS by a mathematical symbol:

$$h = \{\langle x, h(x) \rangle \mid x \in X\}$$

(2)

where $h(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set $h$. For convenience, Xia and Xu [15] called $h = h(x)$ a hesitant fuzzy element (HFE).

## 2.3 Hesitant Fuzzy Linguistic Term Sets (HFLTSs)

Hesitant fuzzy sets [5] were presented to describe and manage the situations where experts may consider several numerical values at the same time to define the membership of an element. However, similar situations may occur in the qualitative setting in which experts hesitate among several linguistic terms instead of numerical values to assess a linguistic variable. For such situations, based on the fuzzy linguistic approach and the hesitant fuzzy sets, Rodríguez et al. [30] introduced the concept of a hesitant fuzzy linguistic term set (HFLTS) as follow:

**Definition 2.2 [30].** Let $S$ be a linguistic term set, $S = \{s_i | i = 0, 1, 2, \cdots, g\}$, a hesitant fuzzy linguistic term set (HFLTS), $H_S$, is an ordered finite subset of consecutive linguistic terms of $S$.

In the following discussions, $H_S$ is simply denoted without ambiguity as $H$. Without the loss of generality, we assume that the elements in a HFLTE are arranged in an increasing order. Let $l_H$ denote the number of linguistic terms in $H$.

**Example 2.1** Let $S$ be a linguistic term set, i.e.,

$$S = \begin{cases} s_0 : \text{extremely poor}, \ s_1 : \text{very poor}, \ s_2 : \text{poor}, \ s_3 : \text{slightly poor}, \ s_4 : \text{fair}, \\ s_5 : \text{slightly good}, \ s_6 : \text{good}, \ s_7 : \text{very good}, \ s_8 : \text{extremely good} \end{cases}.$$

We give two hesitant fuzzy linguistic term sets (HFLTEs) $H_1$ and $H_2$ as follows:

$$H_1 = \{s_3 : \text{slightly poor}, \ s_4 : \text{fair}\},$$
$$H_2 = \{s_4 : \text{fair}, \ s_5 : \text{slightly good}, \ s_6 : \text{good}, \ s_7 : \text{very good}\}.$$

Given two HFLTSs $H_1$ and $H_2$, in most cases, $l_{H_1} \neq l_{H_2}$. To operate correctly between HFLTSs, it is required that they have the same length. To address this issue, Zhu and Xu [42] extended the shorter HFLTS until both of them have the same length by the following method:

**Definition 2.3 [42].** Assume a HFLTS, $H = \{H^t | t = 1, 2, \cdots, l_H\}$, let $H^+$ and $H^-$ be the maximum and minimum linguistic terms in $H$ respectively, and $\varsigma$ ($0 \leq \varsigma \leq 1$) be an optimized parameter, then we call $\bar{H} = \varsigma H^+ + (1 - \varsigma) H^-$ an added linguistic terms.

We can add linguistic terms to a HFLTS by using $\varsigma$, where $\varsigma$ is provided by the decision maker (DM) according to his/her risk preference. When $\varsigma = 1$, $\varsigma = 0$ or $\varsigma = \dfrac{1}{2}$, we have $\bar{H} = H^+$, $\bar{H} = H^-$ or $\bar{H} = \dfrac{1}{2}\left(H^+ + H^-\right)$, respectively, which indicate that the DM's risk preference is risk-seeking, risk-averse or risk-neutral.

**Example 2.2** Let $H_1$ and $H_2$ are two HFLTSs shown in Example 2.1. Clearly, $l_{H_1} = 2$, $l_{H_2} = 4$ and $l_{H_1} < l_{H_2}$. Assume that the DM is a risk-seeker, i.e., $\varsigma = 1$. Then $H_1$ can be extended to the following form:

$$H_1 = \left\{ s_3 : \text{slightly poor}, \ s_4 : \text{fair}, \ s_4 : \text{fair}, \ s_4 : \text{fair} \right\}.$$

Based on Eq. (2), we define the distance between two HFLTSs, $H_1$, $H_2$, as below:

**Definition 2.4.** Let $S$ be a linguistic term set, and $H_1$, $H_2$ ($l_{H_1} = l_{H_2} = l_H$) be two HFLTSs on $S$, then the distance between them is defined as:

$$d\left(H_1, H_2\right) = \frac{\sum\limits_{t=1}^{l_H} d\left(H_1^t, H_2^t\right)}{l_H} = \frac{\sum\limits_{t=1}^{l_H} \left| I\left(H_1^t\right) - I\left(H_2^t\right) \right|}{g \cdot l_H} \tag{3}$$

where $H_1 = \left\{ H_1^t \middle| t = 1, 2, \cdots, l_H \right\}$ and $H_2 = \left\{ H_2^t \middle| t = 1, 2, \cdots, l_H \right\}$.

## 3. A NOVEL METHOD FOR MULTIPLE ATTRIBUTE GROUP DECISION MAKING WITH HESITANT FUZZY LINGUISTIC INFORMATION

### 3.1 Problem Description

First, a multiple attribute group decision making (MAGDM) problem with hesitant fuzzy linguistic information can be summarized as follows: Let $X = \left\{ x_1, x_2, \cdots, x_m \right\}$ be a set of $m$ alternatives, $C = \left\{ c_1, c_2, \ldots, c_n \right\}$ be a collection of $n$ attributes, whose weight vector is $w = \left( w_1, w_2, \cdots, w_n \right)^T$, with $w_j \in [0,1]$, $j = 1, 2, \cdots, n$, and $\sum\limits_{j=1}^{n} w_j = 1$, and let $D = \left\{ d_1, d_2, \cdots, d_p \right\}$ is a set of $p$ decision makers, whose weight vector is $\omega = \left( \omega_1, \omega_2, \cdots, \omega_p \right)^T$, with $\omega_k \in [0,1]$, $k = 1, 2, \cdots, p$, and $\sum\limits_{k=1}^{p} \omega_k = 1$. Let $\mathcal{A}^{(k)} = \left( A_{ij}^{(k)} \right)_{m \times n}$ be a

hesitant fuzzy linguistic decision matrix, where $A_{ij}^{(k)} = \left\{ \left( A_{ij}^{(k)} \right)^t \middle| t = 1, 2, \cdots, l_{A_{ij}^{(k)}} \right\} \in \tilde{H}$ is a HFLTS, which is a set of all the linguistic terms that the alternative $x_i \in X$ satisfies the attribute $c_j \in C$, given by the decision maker $d_k \in D$.

In general, there are benefit attributes (i.e., the bigger the attribute values the better) and cost attributes (i.e., the smaller the attribute values the better) in a MAGDM problem. For such cases, we need to transform the hesitant fuzzy linguistic decision matrices $\mathcal{A}^{(k)} = \left( A_{ij}^{(k)} \right)_{m \times n}$ ($k = 1, 2, \cdots, p$) into the normalized hesitant fuzzy linguistic decision matrix $\mathcal{B}^{(k)} = \left( B_{ij}^{(k)} \right)_{m \times n}$ ($k = 1, 2, \cdots, p$) by the following equation:

$$B_{ij}^{(k)} = \begin{cases} A_{ij}^{(k)}, & \text{for benefit attribute } c_j \\ \left( A_{ij}^{(k)} \right)^c, & \text{for cost attribute } c_j \end{cases}, \quad i = 1, 2, \cdots, m, \ j = 1, 2, \cdots, n, \ k = 1, 2, \cdots, p, \quad (4)$$

where $\left( A_{ij}^{(k)} \right)^c$ is the complement of $A_{ij}^{(k)}$, such that $\left( A_{ij}^{(k)} \right)^c = \left\{ \text{neg} \left( \left( A_{ij}^{(k)} \right)^t \right) \middle| t = 1, 2, \cdots, l_{A_{ij}^{(k)}} \right\}$.

In most situations, it is noted that the numbers of linguistic terms in different HFLTSs $B_{ij}^{(k)}$ of $\mathcal{B}^{(k)}$ ($k = 1, 2, \cdots, p$) are different. In order to more accurately calculate the distance between these HFLTSs, we should extend the shorter ones until all of them have the same length. Let $l = \max \left\{ l_{B_{ij}^{(k)}} \middle| i = 1, 2, \cdots, m, \ j = 1, 2, \cdots, n, \ k = 1, 2, \cdots, p \right\}$. By the regulations mentioned in Definition 2.3, we transform the hesitant fuzzy linguistic decision matrices $\mathcal{B}^{(k)} = \left( B_{ij}^{(k)} \right)_{m \times n}$ ($k = 1, 2, \cdots, p$) into the corresponding hesitant fuzzy linguistic decision matrices $\mathcal{H}^{(k)} = \left( H_{ij}^{(k)} \right)_{m \times n}$ ($k = 1, 2, \cdots, p$), such that $l_{H_{ij}^{(k)}} = l$ for all $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, and $k = 1, 2, \cdots, p$.

### 3.2 A Quadratic Programming Model for Determining the Weights of Decision Makers

First, we aggregate the individual hesitant fuzzy linguistic decision matrices $\mathcal{H}^{(k)} = \left( H_{ij}^{(k)} \right)_{m \times n}$ ($k = 1, 2, \cdots, p$) into the group hesitant fuzzy linguistic decision matrix $\mathcal{H} = \left( H_{ij} \right)_{m \times n}$, where

$$H_{ij} = \overset{p}{\underset{k=1}{\oplus}} \left( \omega_k H_{ij}^{(k)} \right) = \left\{ \overset{p}{\underset{k=1}{\oplus}} \omega_k \left( H_{ij}^{(k)} \right)^t \middle| t = 1, 2, \cdots, l \right\}$$

(5)

In general, the smaller the deviation between the individual decision information and the group decision information, the larger the consensus between the individual decision information and the group decision information, the closer that the individual decision information is to the group decision information, the more reliable the individual decision information. Therefore, the criterion of determining the optimal weights of decision makers is to minimize the deviation measure between the individual hesitant fuzzy linguistic decision matrices and the group hesitant fuzzy linguistic decision matrix.

In the following, motivated by Xu and Cai [43,44], we consider the issue how to determine the weights of decision makers, which can be discussed in the following two cases:

(1) If all
$$\mathcal{H}^{(k)} = \left( H_{ij}^{(k)} \right)_{m \times n} = \left( \left\{ \left( H_{ij}^{(k)} \right)^t \middle| t = 1, 2, \cdots, l \right\} \right)_{m \times n} \ (\ k = 1, 2, \cdots, p\ )$$ are the same as
$$\mathcal{H} = \left( H_{ij} \right)_{m \times n} = \left( \left\{ H_{ij}^t \middle| t = 1, 2, \cdots, l \right\} \right)_{m \times n},$$ then it is reasonable to assign the decision makers $d_k$

( $k = 1, 2, \cdots, p$ ) the same weights $\dfrac{1}{p}$ .

(2) If all $\mathcal{H}^{(k)}$ ( $k = 1, 2, \cdots, p$ ) are not the same as $\mathcal{H}$, then we introduce the deviation variables

$$e_{ij}^{(k)} = d\left( H_{ij}^{(k)}, H_{ij} \right) = \frac{\sum_{t=1}^{l} \left| I\left( \left( H_{ij}^{(k)} \right)^t \right) - I\left( H_{ij}^t \right) \right|}{g \cdot l} = \frac{\sum_{t=1}^{l} \left| I\left( \left( H_{ij}^{(k)} \right)^t \right) - I\left( \bigoplus_{q=1}^{p} \omega_q \left( H_{ij}^{(q)} \right)^t \right) \right|}{g \cdot l}$$

$$= \frac{\sum_{t=1}^{l} \left| I\left( \left( H_{ij}^{(k)} \right)^t \right) - \sum_{q=1}^{p} \omega_q I\left( \left( H_{ij}^{(q)} \right)^t \right) \right|}{g \cdot l} \tag{6}$$

for all $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, $k = 1, 2, \cdots, p$,

and then define the square deviations among all $\mathcal{H}^{(k)}$ ( $k = 1, 2, \cdots, p$ ) and $\mathcal{H}$ as below:

$$e(\omega) = \frac{1}{mnpgl} \sum_{k=1}^{p} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{t=1}^{l} \left( I\left( \left( H_{ij}^{(k)} \right)^t \right) - \sum_{q=1}^{p} \omega_q I\left( \left( H_{ij}^{(q)} \right)^t \right) \right)^2 \tag{7}$$

Based on the viewpoint of maximizing group consensus, we can construct the quadratic programming model as follows:

$$\begin{cases} \min e(\omega) = \min \dfrac{1}{mnpgl} \sum_{k=1}^{p} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{t=1}^{l} \left( I\left( \left( H_{ij}^{(k)} \right)^t \right) - \sum_{q=1}^{p} \omega_q I\left( \left( H_{ij}^{(q)} \right)^t \right) \right)^2 \\ \text{s.t.} \quad \omega_k > 0, \quad k = 1, 2, \cdots, p, \quad \sum_{k=1}^{p} \omega_k = 1 \end{cases} \tag{M-1}$$

We can derive the solution to the model (M-1) by the following procedures:
We first construct the Lagrange function:

$$L(\omega,\lambda) = \frac{1}{mnpgl} \sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left( I\left(\left(H_{ij}^{(k)}\right)^t\right) - \sum_{q=1}^{p}\omega_q I\left(\left(H_{ij}^{(q)}\right)^t\right)\right)^2 - 2\lambda\left(\sum_{k=1}^{p}\omega_k - 1\right)$$

(8)

where $\lambda$ is the Lagrange multiplier.

Differentiate (8) with respect to $\omega_h$ ( $h=1,2,\cdots,p$ ), and set these partial derivatives equal to zero, then we have the following equations:

$$\frac{dL(\omega,\lambda)}{d\omega_h} = -\frac{1}{mnpgl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l} 2\left( I\left(\left(H_{ij}^{(k)}\right)^t\right) - \sum_{q=1}^{p}\omega_q I\left(\left(H_{ij}^{(q)}\right)^t\right)\right)\left( I\left(\left(H_{ij}^{(h)}\right)^t\right)\right) - 2\lambda = 0$$

(9)

i.e.

$$\left(\sum_{q=1}^{p}\omega_q\left(\frac{1}{mngl}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l} I\left(\left(H_{ij}^{(h)}\right)^t\right) I\left(\left(H_{ij}^{(q)}\right)^t\right)\right)\right) - \frac{1}{mnpgl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left( I\left(\left(H_{ij}^{(k)}\right)^t\right) I\left(\left(H_{ij}^{(h)}\right)^t\right)\right) - \lambda = 0$$

(10)

which can be rewritten in matrix form as:

$$D\omega - A - \lambda E = 0$$

(11)

where $E = (1,1,\cdots,1)^T$, and

$$A = \frac{1}{mnpgl}\begin{pmatrix} \sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left( I\left(\left(H_{ij}^{(k)}\right)^t\right) I\left(\left(H_{ij}^{(1)}\right)^t\right)\right) \\ \sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left( I\left(\left(H_{ij}^{(k)}\right)^t\right) I\left(\left(H_{ij}^{(2)}\right)^t\right)\right) \\ \cdots \\ \sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left( I\left(\left(H_{ij}^{(k)}\right)^t\right) I\left(\left(H_{ij}^{(p)}\right)^t\right)\right) \end{pmatrix}_{p\times 1}$$

(12)

Since $\sum_{k=1}^{p}\omega_k = 1$ can be rewritten as $E^T\omega = 1$, from which and (11), we obtain

$$\lambda = \frac{1 - E^T D^{-1} A}{E^T D^{-1} E}$$

(13)

and

$$\omega = \frac{D^{-1}E\left(1 - E^T D^{-1} A\right)}{E^T D^{-1} E} + D^{-1} A$$

(14)

where $\omega$ is the weight vector of decision makers $d_k$ ( $k = 1, 2, \cdots, p$ ).

## 3.3 Obtaining the Optimal Weights of Attributes by the Maximizing Deviation Method

Because many practical group decision making problems are complex and uncertain, and human thinking is inherently subjective, the information about attribute weights is usually incomplete. For convenience, let $\Delta$ be a set of the known weight information [45-48], where $\Delta$ can be constructed by the following forms, for $i \neq j$ :

**Form 1.** A weak ranking: $\{w_i \geq w_j\}$ ;

**Form 2.** A strict ranking: $\{w_i - w_j \geq \alpha_i\}$ ( $\alpha_i > 0$ );

**Form 3.** A ranking of differences: $\{w_i - w_j \geq w_k - w_l\}$ , for $j \neq k \neq l$ ;

**Form 4.** A ranking with multiples: $\{w_i \geq \alpha_i w_j\}$ ( $0 \leq \alpha_i \leq 1$ );

**Form 5.** An interval form: $\{\alpha_i \leq w_i \leq \alpha_i + \varepsilon_i\}$ ( $0 \leq \alpha_i \leq \alpha_i + \varepsilon_i \leq 1$ ).

The maximizing deviation method was proposed by Wang [49] to estimate the attribute weights in MADM problems with numerical information. According to Wang [49], if the performance values of all the alternatives have small differences under an attribute, it shows that such an attribute plays a less important role in choosing the best alternative and should be assigned a smaller weight. On the contrary, if an attribute makes the performance values of all the alternatives have obvious differences, then such an attribute plays a much important role in choosing the best alternative and should be assigned a larger weight. Especially, if all available alternatives score about equally with respect to a given attribute, then such an attribute will be judged unimportant by most decision makers and should be assigned a very small weight. Wang [49] suggests that zero weight should be assigned to the attribute of this kind.

In what follows, based on the maximizing deviation method, we construct an optimization model to determine the optimal relative weights of attributes under hesitant fuzzy linguistic environment. It is noted that our optimization model is similar to the minimizing deviations models proposed by Xu [50] for solving MADM problems with preference information on alternatives in uncertain linguistic setting.

For the attribute $c_j \in C$ , the deviation of the alternative $x_i \in X$ to all the other alternatives with respect to the decision maker $d_k \in D$ can be defined as below:

$$D_{ij}^{(k)}(w) = \sum_{q=1}^{m} d\left(H_{ij}^{(k)}, H_{qj}^{(k)}\right) w_j = \frac{\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l}\left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right)\right| w_j}{g \cdot l},$$

$$i = 1, 2, \cdots, m \,,\; j = 1, 2, \cdots, n \,,\; k = 1, 2, \cdots, p \tag{15}$$

Let

$$D_j^{(k)}(w) = \sum_{i=1}^{m} D_{ij}^{(k)}(w) = \sum_{i=1}^{m}\sum_{q=1}^{m} d\left(H_{ij}^{(k)}, H_{qj}^{(k)}\right) w_j = \frac{\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l}\left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right)\right| w_j}{g \cdot l}$$

$$j = 1, 2, \cdots, n \,,\; k = 1, 2, \cdots, p \tag{16}$$

then $D_j^{(k)}(w)$ represents the deviation value of all alternatives to other alternatives for the attribute $c_j \in C$ with respect to the decision maker $d_k \in D$.

Further, let

$$D(w) = \sum_{k=1}^{p} \omega_k \left( \sum_{j=1}^{n} D_j^{(k)}(w) \right) = \sum_{k=1}^{p} \omega_k \left( \sum_{j=1}^{n}\sum_{i=1}^{m} D_{ij}^{(k)}(w) \right) = \sum_{k=1}^{p} \omega_k \left( \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q=1}^{m} d\left(H_{ij}^{(k)}, H_{qj}^{(k)}\right) w_j \right)$$

$$= \frac{\sum\limits_{k=1}^{p} \omega_k \left( \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l}\left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right)\right| w_j \right)}{g \cdot l} \tag{17}$$

then $D(w)$ represents the deviation value of all alternatives to other alternatives for all the attributes with respect to all the decision makers.

Based on the above analysis, we can construct a non-linear programming model to select the weight vector $w$ by maximizing $D(w)$, as follows:

$$\begin{cases} \max D(w) = \max \dfrac{\sum\limits_{k=1}^{p} \omega_k \left( \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l}\left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right)\right| w_j \right)}{g \cdot l} \\[6mm] \text{s.t.} \quad w_j \geq 0, \; j = 1, 2, \cdots, n, \quad \sum\limits_{j=1}^{n} w_j^2 = 1 \end{cases} \tag{M-2}$$

To solve this model, we construct the Lagrange function:

$$L(w,\lambda) = \frac{\sum\limits_{k=1}^{p} \omega_k \left( \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l} \left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right) \right| w_j \right)}{g \cdot l} + \frac{\lambda}{2}\left( \sum\limits_{j=1}^{n} w_j^2 - 1 \right)$$

(18)

where $\lambda$ is the Lagrange multiplier.

Differentiating Eq. (18) with respect to $w_j$ ( $j=1,2,\cdots,n$ ) and $\lambda$, and setting these partial derivatives equal to zero, then the following set of equations is obtained:

$$\frac{\partial L}{\partial w_j} = \frac{\sum\limits_{k=1}^{p}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l} \left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right) \right| \omega_k}{g \cdot l} + \lambda w_j = 0$$

(19)

$$\frac{\partial L}{\partial \lambda} = \frac{1}{2}\left( \sum\limits_{j=1}^{n} w_j^2 - 1 \right) = 0$$

(20)

It follows from Eq. (20) that

$$w_j = \frac{-\sum\limits_{k=1}^{p}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l} \left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right) \right| \omega_k}{\lambda g \cdot l}$$

(21)

Putting Eq. (21) into Eq. (19), we get

$$\lambda = \frac{-\sqrt{\sum\limits_{j=1}^{n}\left( \sum\limits_{k=1}^{p}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l} \left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right) \right| \omega_k \right)^2}}{g \cdot l}$$

(22)

Then combining Eqs. (21) and (22), we have

$$w_j = \frac{\sum\limits_{k=1}^{p}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l} \left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right) \right| \omega_k}{\sqrt{\sum\limits_{j=1}^{n}\left( \sum\limits_{k=1}^{p}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l} \left| I\left(\left(H_{ij}^{(k)}\right)^t\right) - I\left(\left(H_{qj}^{(k)}\right)^t\right) \right| \omega_k \right)^2}}$$

(23)

By normalizing $w_j$ ( $j=1,2,\cdots,n$ ), we make their sum into a unit, and get

$$w_j^* = \frac{w_j}{\sum\limits_{j=1}^{n} w_j} = \frac{\sum\limits_{k=1}^{p}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l}\left|I\left(\left(H_{ij}^{(k)}\right)^{t}\right) - I\left(\left(H_{qj}^{(k)}\right)^{t}\right)\right|\omega_k}{\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{p}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l}\left|I\left(\left(H_{ij}^{(k)}\right)^{t}\right) - I\left(\left(H_{qj}^{(k)}\right)^{t}\right)\right|\omega_k}$$

(24)

which can be considered as the optimal weight vector of attributes.

However, it is noted that there are practical situations in which the information about the weight vector is not completely unknown but partially known. For such cases, we establish the following constrained optimization model:

$$\begin{cases} \max D(w) = \max \dfrac{\sum\limits_{k=1}^{p}\omega_k\left(\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{m}\sum\limits_{t=1}^{l}\left|I\left(\left(H_{ij}^{(k)}\right)^{t}\right) - I\left(\left(H_{qj}^{(k)}\right)^{t}\right)\right|w_j\right)}{g \cdot l} \\[4mm] \text{s.t.} \quad w \in \Delta, \quad w_j \geq 0, \ j = 1, 2, \cdots, n, \quad \sum\limits_{j=1}^{n} w_j = 1 \end{cases}$$

(M-3)

It is noted that the model (M-3) is a linear programming model that can be solved using the MATLAB mathematics software package. Suppose that the optimal solution to (M-2) is $w = \left(w_1, w_2, \cdots, w_n\right)^T$, which can be considered as the weight vector of attributes.

## 3.4 Extended TOPIS Method for the MAGDM with Hesitant Fuzzy Linguistic Information

TOPSIS method, initially introduced by Hwang and Yoon [51], is a widely used method for dealing with MADM problems, which focuses on choosing the alternative with the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). In the following, based on the above analysis, we shall extend the TOPIS method to the MAGDM problems under hesitant fuzzy linguistic environments, in which the information about the weights of decision makers is unknown, the information about attribute weights is incompletely known or completely unknown, and the attribute values are given in the form of HFLTSs.

The flowchart of the extended TOPIS method is provided in Fig. 1. The extended method consists of the following steps:

**Step 1.** For a MAGDM problem, the decision maker $d_k \in D$ constructs the hesitant fuzzy linguistic decision matrix $\mathcal{A}^{(k)} = \left(A_{ij}^{(k)}\right)_{m \times n}$, where $A_{ij}^{(k)}$ is a HFLTS, given by the DM $d_k \in D$, for the alternative $x_i \in X$ with respect to the attribute $c_j \in C$. Utilize Eq. (4) to transform the hesitant fuzzy linguistic decision matrices $\mathcal{A}^{(k)} = \left(A_{ij}^{(k)}\right)_{m \times n}$ ( $k = 1, 2, \cdots, p$ ) into the normalized hesitant fuzzy linguistic decision matrices $\mathcal{H}^{(k)} = \left(H_{ij}^{(k)}\right)_{m \times n}$ ( $k = 1, 2, \cdots, p$ ).

**Step 2.** If the information about the weights of decision makers is unknown, then we use Eq. (14) to obtain the weights of decision makers.

**Step 3.** If the information about the attribute weights is completely unknown, then we use Eq. (24) to obtain the attribute weights; if the information about the attribute weights is partly known, then we solve the model (M-2) to obtain the attribute weights.

**Step 4.** Determine the hesitant fuzzy linguistic positive ideal solution (PIS) $X_+^{(k)} = \left\{ H_{+1}^{(k)}, H_{+2}^{(k)}, \cdots, H_{+n}^{(k)} \right\}$ and the hesitant fuzzy linguistic negative ideal solution (NIS) $X_-^{(k)} = \left\{ H_{-1}^{(k)}, H_{-2}^{(k)}, \cdots, H_{-n}^{(k)} \right\}$ for each decision maker $d_k$ by the following equations:

$$H_{+j}^{(k)} = \max_i \left\{ H_{ij}^{(k)} \right\} = \left\{ \max_i \left\{ \left( H_{ij}^{(k)} \right)^t \right\} \middle| t = 1, 2, \cdots, l \right\} \qquad j = 1, 2, \cdots, n \tag{25}$$

$$H_{-j}^{(k)} = \min_i \left\{ H_{ij}^{(k)} \right\} = \left\{ \min_i \left\{ \left( H_{ij}^{(k)} \right)^t \right\} \middle| t = 1, 2, \cdots, l \right\} \qquad j = 1, 2, \cdots, n \tag{26}$$

**Step 5.** Calculate the separation measures $d_{+i}^{(k)}$ of each alternative $x_i$ from the hesitant fuzzy linguistic PIS $X_+^{(k)}$ of the decision maker $d_k$ as:

$$d_{+i}^{(k)} = \sum_{j=1}^n w_j d\left( H_{ij}^{(k)}, H_{+j}^{(k)} \right) = \frac{\sum_{j=1}^n \sum_{t=1}^l w_j \left| I\left( \left( H_{ij}^{(k)} \right)^t \right) - I\left( \left( H_{+j}^{(k)} \right)^t \right) \right|}{g \cdot l} \tag{27}$$

In a similar way, calculate the separation measures $d_{-i}^{(k)}$ of each alternative $x_i$ from the hesitant fuzzy linguistic NIS $X_-^{(k)}$ of the decision maker $d_k$ as:

$$d_{-i}^{(k)} = \sum_{j=1}^n w_j d\left( H_{ij}^{(k)}, H_{-j}^{(k)} \right) = \frac{\sum_{j=1}^n \sum_{t=1}^l w_j \left| I\left( \left( H_{ij}^{(k)} \right)^t \right) - I\left( \left( H_{-j}^{(k)} \right)^t \right) \right|}{g \cdot l} \tag{28}$$

**Step 6.** Calculate the relative closeness coefficient of each alternative $x_i$ to the hesitant fuzzy linguistic PIS $X_+^{(k)}$ of the decision maker $d_k$ as:

$$C_i^{(k)} = \frac{d_{-i}^{(k)}}{d_{+i}^{(k)} + d_{-i}^{(k)}} \tag{29}$$

After calculating the $C_i^{(k)}$ for each decision maker $d_k$ ( $k = 1, 2, \cdots, p$ ), we then form the

relative-closeness coefficient matrix as below:

$$C = \begin{pmatrix} C_1^{(1)} & C_1^{(2)} & \cdots & C_1^{(p)} \\ C_2^{(1)} & C_2^{(2)} & \cdots & C_2^{(p)} \\ \cdots & \cdots & \cdots & \cdots \\ C_m^{(1)} & C_m^{(2)} & \cdots & C_m^{(p)} \end{pmatrix}_{m \times p}$$

(30)

Steps 4-6 extended the standard TOPSIS to hesitant fuzzy linguistic environments and therefore can be called the hesitant fuzzy linguistic TOPSIS. From this stage on our method continues by applying the standard TOPSIS to the relative-closeness coefficient decision matrix in order to identify the group positive ideal solution.

**Step 7.** Identify the group positive ideal solution (GPIS) and group negative ideal solution (GNIS), respectively as follows:

$$X_+^G = \left\{ \max_i \left\{ C_i^{(1)} \right\}, \max_i \left\{ C_i^{(2)} \right\}, \cdots, \max_i \left\{ C_i^{(p)} \right\} \right\}$$

(31)

$$X_-^G = \left\{ \min_i \left\{ C_i^{(1)} \right\}, \min_i \left\{ C_i^{(2)} \right\}, \cdots, \min_i \left\{ C_i^{(p)} \right\} \right\}$$

(32)

**Step 8.** Calculate the separation measures $d_{+i}^G$ and $d_{-i}^G$ of each alternative $x_i$ from the group positive ideal solution $X_+^G$ and the group negative ideal solution $X_-^G$, respectively, as follows:

$$d_{+i}^G = \sum_{k=1}^{p} \omega_k d \left( C_i^{(k)}, \max_i \left\{ C_i^{(k)} \right\} \right) = \sum_{k=1}^{p} \omega_k \left| C_i^{(k)} - \left( \max_i \left\{ C_i^{(k)} \right\} \right) \right|$$

(33)

$$d_{-i}^G = \sum_{k=1}^{p} \omega_k d \left( C_i^{(k)}, \min_i \left\{ C_i^{(k)} \right\} \right) = \sum_{k=1}^{p} \omega_k \left| C_i^{(k)} - \left( \min_i \left\{ C_i^{(k)} \right\} \right) \right|$$

(34)

**Step 9.** Calculate the group relative-closeness coefficient $C_i^G$ of each alternative $x_i$ to group positive ideal solution $d_{+i}^G$ as:

$$C_i^G = \frac{d_{-i}^G}{d_{+i}^G + d_{-i}^G}$$

(35)

**Step 10.** Rank the alternatives $x_i$ ( $i = 1, 2, \cdots, m$ ) according to the group relative-closeness coefficients $C_i^G$ ( $i = 1, 2, \cdots, m$ ) and then select the most desirable one(s). The larger the value of $C_i^G$, the more different between $x_i$ and the group negative ideal object $d_{-i}^G$, while the more similar between $x_i$ and the group positive ideal object $d_{+i}^G$. Therefore, the alternative(s) with the maximum group relative-closeness coefficient should be chosen as the optimal one(s).
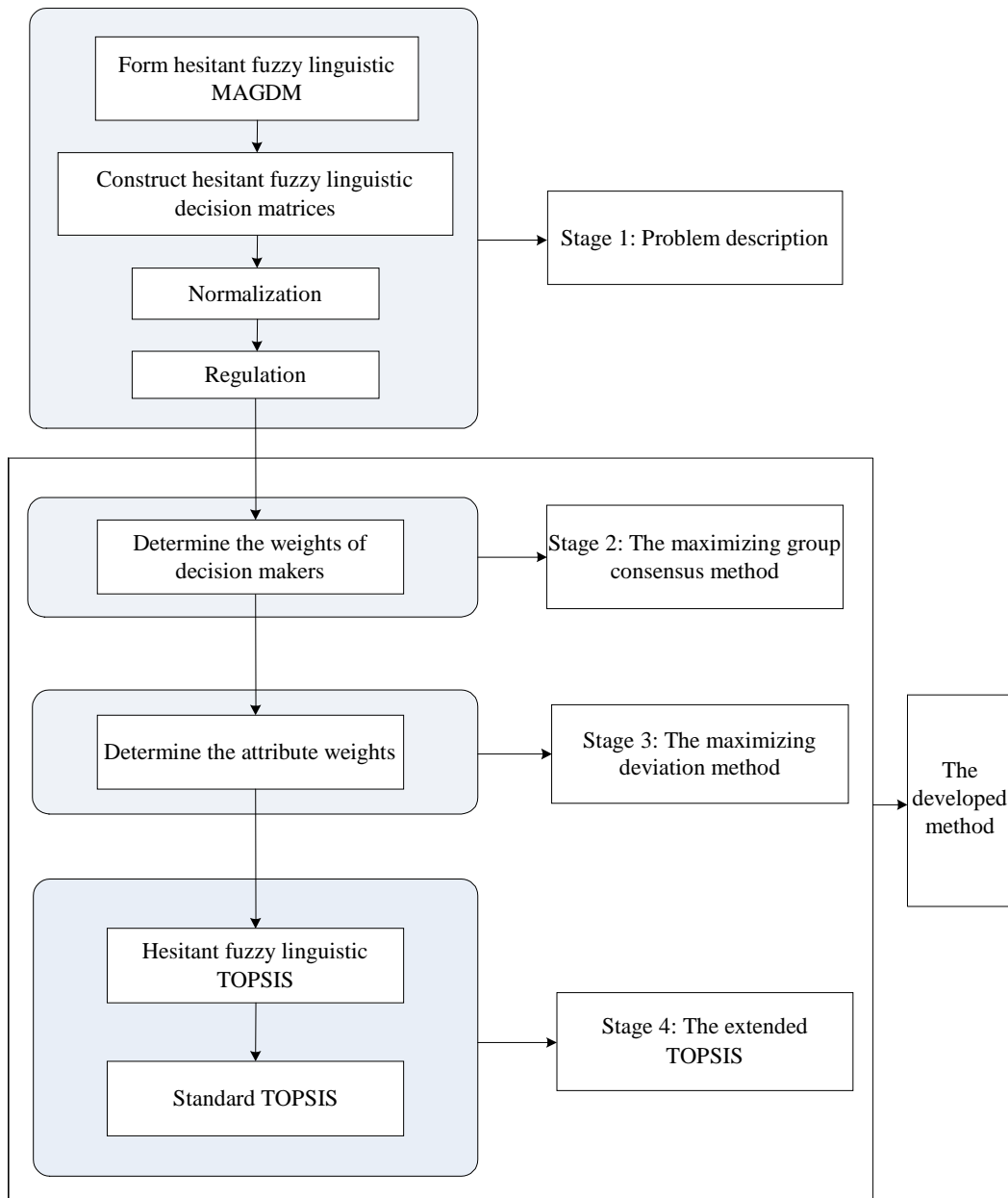
**Fig. 1. The flowchart of the developed method**

It is noted that some formulas such as (25) and (26) and the similar TOPSIS approach with hesitant fuzzy linguistic information can be found in Ref. [52]. However, Ref. [52] focuses on the MADM problems with only a hesitant fuzzy linguistic decision matrix. However, in real-life, due to the increasing complexity of socio-economic environment, it is less and less possible for a single decision maker to consider all relevant aspects of the problem. Therefore, many organizations employ groups to make decision, which is called as group decision making

(GDM). Our method gives a TOPIS based procedure to solve a MAGDM problem under hesitant fuzzy linguistic environments with several hesitant fuzzy linguistic decision matrices. The TOPSIS method proposed in Ref. [52] included a stage, which we call the hesitant fuzzy linguistic TOPSIS; while the extended TOPSIS proposed by our method includes two stages. The first stage is called the hesitant fuzzy linguistic TOPSIS, which can be used to calculate the individual relative closeness coefficient of each alternative to the individual hesitant fuzzy linguistic PIS. The second stage is the standard TOPSIS, which is used to calculate the group relative-closeness coefficient of each alternative to group PIS and select the optimal one with the maximum group relative-closeness coefficient.

## 4. ILLUSTRATED EXAMPLE

In this section, a numerical example is used to demonstrate the applicability and the effectiveness of our method under hesitant fuzzy linguistic environment.

**Example 4.1** Let us suppose an investment company, which wants to invest a sum of money in the best option (adapted from [35,53]). There is a panel with five possible alternatives in which to invest the money: (1) $x_1$ is a car industry; (2) $x_2$ is a food company; (3) $x_3$ is a computer company; (4) $x_4$ is an arms company; (5) $x_5$ is a TV company. The investment company must make a decision according to the following four attributes: (1) $c_1$ is the risk analysis; (2) $c_2$ is the growth analysis; (3) $c_3$ is the social–political impact analysis; (4) $c_4$ is the environmental impact analysis. Suppose that five possible candidates $x_i$ $(i = 1,2,3,4,5)$ are to be evaluated using the linguistic term set

$$S = \left\{ \begin{array}{l} s_0 = \text{extremely poor}, \ s_1 = \text{very poor}, \ s_2 = \text{poor}, \ s_3 = \text{slightly poor}, \ s_4 = \text{fair}, \\ s_5 = \text{slightly good}, \ s_6 = \text{good}, \ s_7 = \text{very good}, \ s_8 = \text{extremely good} \end{array} \right\}$$

by three decision makers $d_k$ ($k = 1,2,3$) under the above four attributes $c_j$ ($j = 1,2,3,4$). The decision makers construct, respectively, three hesitant fuzzy linguistic decision matrices $\mathcal{A}^{(k)} = \left( A_{ij}^{(k)} \right)_{5 \times 4}$ ($k = 1,2,3$) listed in Tables 1-3, where $A_{ij}^{(k)}$ is a HFLTS denoting all the possible linguistic terms for the alternative $x_i$ under the attribute $c_j$. The hierarchical structure of this MAGDM problem is shown in Fig. 2.

**Table 1. Hesitant fuzzy linguistic decision matrix $\mathcal{A}^{(1)}$ provided by the decision maker $d_1$**

| 1 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $\{s_5,s_6,s_7\}$ | $\{s_5,s_6,s_7,s_8\}$ | $\{s_3,s_4,s_5\}$ | $\{s_2,s_3,s_4,s_5\}$ |
| $x_2$ | $\{s_5,s_6\}$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_1,s_2,s_3,s_4\}$ | $\{s_4,s_5,s_6\}$ |
| $x_3$ | $\{s_2,s_3\}$ | $\{s_7,s_8\}$ | $\{s_2,s_3,s_4\}$ | $\{s_5,s_6\}$ |
| $x_4$ | $\{s_4,s_5,s_6,s_7\}$ | $\{s_3,s_4,s_5\}$ | $\{s_7,s_8\}$ | $\{s_1,s_2,s_3\}$ |
| $x_5$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_2,s_3,s_4,s_5\}$ | $\{s_4,s_5,s_6\}$ | $\{s_1,s_2,s_3,s_4\}$ |

**Table 2. Hesitant fuzzy linguistic decision matrix $\mathcal{A}^{(2)}$ provided by the decision maker $d_2$**

| 2 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $\{s_5,s_6,s_7\}$ | $\{s_1,s_2,s_3,s_4\}$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_2,s_3,s_4,s_5\}$ |
| $x_2$ | $\{s_3,s_4,s_5,s_6,s_7\}$ | $\{s_5,s_6\}$ | $\{s_5,s_6,s_7,s_8\}$ | $\{s_1,s_2\}$ |
| $x_3$ | $\{s_7,s_8\}$ | $\{s_2,s_3,s_4\}$ | $\{s_3,s_4,s_5\}$ | $\{s_4,s_5,s_6\}$ |
| $x_4$ | $\{s_4,s_5,s_6,s_7\}$ | $\{s_7,s_8\}$ | $\{s_2,s_3,s_4\}$ | $\{s_1,s_2,s_3\}$ |
| $x_5$ | $\{s_5,s_6,s_7\}$ | $\{s_2,s_3,s_4,s_5\}$ | $\{s_7,s_8\}$ | $\{s_1,s_2,s_3,s_4\}$ |

**Table 3. Hesitant fuzzy linguistic decision matrix $\mathcal{A}^{(3)}$ provided by the decision maker $d_3$**

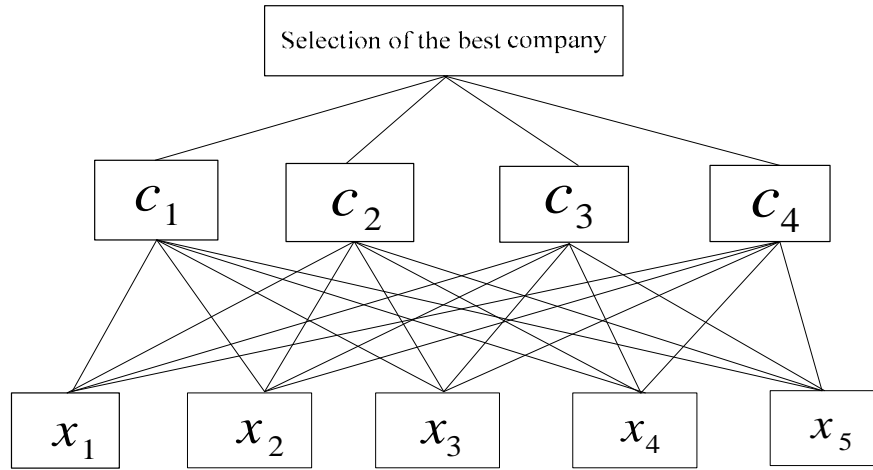| 3 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $\{s_5,s_6,s_7,s_8\}$ | $\{s_6,s_7\}$ | $\{s_3,s_4,s_5\}$ | $\{s_2,s_3,s_4,s_5\}$ |
| $x_2$ | $\{s_4,s_5,s_6\}$ | $\{s_1,s_2,s_3,s_4\}$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_5,s_6\}$ |
| $x_3$ | $\{s_2,s_3\}$ | $\{s_6,s_7,s_8\}$ | $\{s_2,s_3,s_4\}$ | $\{s_5,s_6,s_7,s_8\}$ |
| $x_4$ | $\{s_1,s_2,s_3\}$ | $\{s_5,s_6\}$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_4,s_5,s_6,s_7\}$ |
| $x_5$ | $\{s_7,s_8\}$ | $\{s_1,s_2,s_3,s_4\}$ | $\{s_4,s_5,s_6\}$ | $\{s_5,s_6,s_7\}$ |

**Fig. 2. Hierarchical structure**

In what follows, we utilize the developed method to find the best alternative(s). We now discuss two different cases.

**Case 1.** Assume that the information about the attribute weights is completely unknown; in this case, we use the following steps to get the most desirable alternative(s).

**Step 1.** Considering that all the attributes $c_j$ ( $j=1,2,3,4$ ) are the benefit type attributes, the hesitant fuzzy linguistic decision matrices $\mathcal{A}^{(k)} = \left( A_{ij}^{(k)} \right)_{5\times 4}$ ( $k=1,2,3$ ) do not need normalization. Suppose that all the decision makers (DMs) ( $k=1,2,3$ ) are pessimistic, then we utilize Definition 2.3 to transform the hesitant fuzzy linguistic decision matrices $\mathcal{A}^{(k)} = \left( A_{ij}^{(k)} \right)_{m\times n}$ ( $k=1,2,3$ ) into the hesitant fuzzy linguistic decision matrices $\mathcal{H}^{(k)} = \left( H_{ij}^{(k)} \right)_{m\times n}$ ( $k=1,2,3$ ) (see Tables 4-6), such that $l_{H_{ij}^{(k)}} = 5$ for all $i=1,2,3,4$ , $j=1,2,3,4$ , and $k=1,2,3$ .

**Table 4. Hesitant fuzzy linguistic decision matrix $\mathcal{H}^{(1)}$ provided by the decision maker $d_1$**

| 4 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $\{s_5,s_5,s_5,s_6,s_7\}$ | $\{s_5,s_5,s_6,s_7,s_8\}$ | $\{s_3,s_3,s_3,s_4,s_5\}$ | $\{s_2,s_2,s_3,s_4,s_5\}$ |
| $x_2$ | $\{s_5,s_5,s_5,s_5,s_6\}$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_1,s_1,s_2,s_3,s_4\}$ | $\{s_4,s_4,s_4,s_5,s_6\}$ |
| $x_3$ | $\{s_2,s_2,s_2,s_2,s_3\}$ | $\{s_7,s_7,s_7,s_7,s_8\}$ | $\{s_2,s_2,s_2,s_3,s_4\}$ | $\{s_5,s_5,s_5,s_5,s_6\}$ |
| $x_4$ | $\{s_4,s_4,s_5,s_6,s_7\}$ | $\{s_3,s_3,s_3,s_4,s_5\}$ | $\{s_7,s_7,s_7,s_7,s_8\}$ | $\{s_1,s_1,s_1,s_2,s_3\}$ |
| $x_5$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_2,s_2,s_3,s_4,s_5\}$ | $\{s_4,s_4,s_4,s_5,s_6\}$ | $\{s_1,s_1,s_2,s_3,s_4\}$ |

**Table 5. Hesitant fuzzy linguistic decision matrix $\mathcal{H}^{(2)}$ provided by the decision maker $d_2$**

| 5 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $\{s_5,s_5,s_5,s_6,s_7\}$ | $\{s_1,s_1,s_2,s_3,s_4\}$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_2,s_2,s_3,s_4,s_5\}$ |
| $x_2$ | $\{s_3,s_4,s_5,s_6,s_7\}$ | $\{s_5,s_5,s_5,s_5,s_6\}$ | $\{s_5,s_5,s_6,s_7,s_8\}$ | $\{s_1,s_1,s_1,s_1,s_2\}$ |
| $x_3$ | $\{s_7,s_7,s_7,s_7,s_8\}$ | $\{s_2,s_2,s_2,s_3,s_4\}$ | $\{s_3,s_3,s_3,s_4,s_5\}$ | $\{s_4,s_4,s_4,s_5,s_6\}$ |
| $x_4$ | $\{s_4,s_4,s_5,s_6,s_7\}$ | $\{s_7,s_7,s_7,s_7,s_8\}$ | $\{s_2,s_2,s_2,s_3,s_4\}$ | $\{s_1,s_1,s_1,s_2,s_3\}$ |
| $x_5$ | $\{s_5,s_5,s_5,s_6,s_7\}$ | $\{s_2,s_2,s_3,s_4,s_5\}$ | $\{s_7,s_7,s_7,s_7,s_8\}$ | $\{s_1,s_1,s_2,s_3,s_4\}$ |

**Table 6. Hesitant fuzzy linguistic decision matrix $\mathcal{H}^{(3)}$ provided by the decision maker $d_3$**

| 6 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $\{s_5,s_5,s_6,s_7,s_8\}$ | $\{s_6,s_6,s_6,s_6,s_7\}$ | $\{s_3,s_3,s_3,s_4,s_5\}$ | $\{s_2,s_2,s_3,s_4,s_5\}$ |
| $x_2$ | $\{s_4,s_4,s_4,s_5,s_6\}$ | $\{s_1,s_1,s_2,s_3,s_4\}$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_5,s_5,s_5,s_5,s_6\}$ |
| $x_3$ | $\{s_2,s_2,s_2,s_2,s_3\}$ | $\{s_6,s_6,s_6,s_7,s_8\}$ | $\{s_2,s_2,s_2,s_3,s_4\}$ | $\{s_5,s_5,s_6,s_7,s_8\}$ |
| $x_4$ | $\{s_1,s_1,s_1,s_2,s_3\}$ | $\{s_5,s_5,s_5,s_5,s_6\}$ | $\{s_4,s_5,s_6,s_7,s_8\}$ | $\{s_4,s_4,s_5,s_6,s_7\}$ |
| $x_5$ | $\{s_7,s_7,s_7,s_7,s_8\}$ | $\{s_1,s_1,s_2,s_3,s_4\}$ | $\{s_4,s_4,s_4,s_5,s_6\}$ | $\{s_5,s_5,s_5,s_6,s_7\}$ |

**Step 2.** Utilize the Eq. (14) to get the weights of the decision makers:

$$\omega = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

**Step 3.** Considering that the information about the attribute weights is completely unknown, we utilize the Eq. (24) to get the optimal weight vector of attributes:

$$w = \left(0.2349, 0.2936, 0.2692, 0.2023\right)^T$$

**Step 4.** Utilize Eqs. (25) and (26) to determine the hesitant fuzzy linguistic PIS $X_+^{(k)}$ ( $k=1,2,3$ ) and the hesitant fuzzy linguistic NIS $X_-^{(k)}$ ( $k=1,2,3$ ) for each decision maker $d_k$ ( $k=1,2,3$ ), respectively:

$$X_+^{(1)} = \{\{s_5,s_5,s_6,s_7,s_8\},\{s_7,s_7,s_7,s_7,s_8\},\{s_7,s_7,s_7,s_7,s_8\},\{s_5,s_5,s_5,s_5,s_6\}\}$$
$$X_-^{(1)} = \{\{s_2,s_2,s_2,s_2,s_3\},\{s_2,s_2,s_3,s_4,s_5\},\{s_1,s_1,s_2,s_3,s_4\},\{s_1,s_1,s_1,s_2,s_3\}\}$$
$$X_+^{(2)} = \{\{s_7,s_7,s_7,s_7,s_8\},\{s_7,s_7,s_7,s_7,s_8\},\{s_7,s_7,s_7,s_7,s_8\},\{s_4,s_4,s_4,s_5,s_6\}\}$$
$$X_-^{(2)} = \{\{s_3,s_4,s_5,s_6,s_7\},\{s_1,s_1,s_2,s_3,s_4\},\{s_2,s_2,s_2,s_3,s_4\},\{s_1,s_1,s_1,s_1,s_2\}\}$$

$$X_+^{(3)} = \{\{s_7, s_7, s_7, s_7, s_8\}, \{s_6, s_6, s_6, s_7, s_8\}, \{s_4, s_5, s_6, s_7, s_8\}, \{s_5, s_5, s_6, s_7, s_8\}\}$$
$$X_-^{(3)} = \{\{s_1, s_1, s_1, s_2, s_3\}, \{s_1, s_1, s_2, s_3, s_4\}, \{s_2, s_2, s_2, s_3, s_4\}, \{s_2, s_2, s_3, s_4, s_5\}\}$$

**Step 5.** Utilize Eqs. (27) and (28) to calculate the separation measures $d_{+i}^{(k)}$ and $d_{-i}^{(k)}$ of each alternative $x_i$ of the decision maker $d_k$ :

$d_{+1}^{(1)} = 0.2260$ , $d_{-1}^{(1)} = 0.2975$ , $d_{+2}^{(1)} = 0.2568$ , $d_{-2}^{(1)} = 0.2667$ , $d_{+3}^{(1)} = 0.2722$ , $d_{-3}^{(1)} = 0.2513$ ,
$d_{+4}^{(1)} = 0.2525$ , $d_{-4}^{(1)} = 0.2710$ , $d_{+5}^{(1)} = 0.3160$ , $d_{-5}^{(1)} = 0.2075$ ,
$d_{+1}^{(2)} = 0.3063$ , $d_{-1}^{(2)} = 0.1826$ , $d_{+2}^{(2)} = 0.2576$ , $d_{-2}^{(2)} = 0.2312$ , $d_{+3}^{(2)} = 0.2900$ , $d_{-3}^{(2)} = 0.1989$ ,
$d_{+4}^{(2)} = 0.2894$ , $d_{-4}^{(2)} = 0.1995$ , $d_{+5}^{(2)} = 0.2545$ , $d_{-5}^{(2)} = 0.2344$ ,
$d_{+1}^{(3)} = 0.2007$ , $d_{-1}^{(3)} = 0.3155$ , $d_{+2}^{(3)} = 0.2631$ , $d_{-2}^{(3)} = 0.2531$ , $d_{+3}^{(3)} = 0.2612$ , $d_{-3}^{(3)} = 0.2550$ ,
$d_{+4}^{(3)} = 0.2411$ , $d_{-4}^{(3)} = 0.2751$ , $d_{+5}^{(3)} = 0.2238$ , $d_{-5}^{(3)} = 0.2924$ .

**Step 6.** Utilize Eq. (29) to calculate the relative closeness coefficient $C_i^{(k)}$ of each alternative $x_i$ to the hesitant fuzzy linguistic PIS $X_+^{(k)}$ of the decision maker $d_k$ as
$C_1^{(1)} = 0.5683$ , $C_2^{(1)} = 0.5095$ , $C_3^{(1)} = 0.4800$ , $C_4^{(1)} = 0.5176$ , $C_5^{(1)} = 0.3964$ ,
$C_1^{(2)} = 0.3735$ , $C_2^{(2)} = 0.4730$ , $C_3^{(2)} = 0.4069$ , $C_4^{(2)} = 0.4081$ , $C_5^{(2)} = 0.4794$ ,
$C_1^{(3)} = 0.6113$ , $C_2^{(3)} = 0.4902$ , $C_3^{(3)} = 0.4940$ , $C_4^{(3)} = 0.5329$ , $C_5^{(3)} = 0.5665$ .

Then, we construct the relative-closeness coefficient matrix as below:

$$C = \begin{pmatrix} 0.5683 & 0.3735 & 0.6113 \\ 0.5095 & 0.4730 & 0.4902 \\ 0.4800 & 0.4069 & 0.4940 \\ 0.5176 & 0.4081 & 0.5329 \\ 0.3964 & 0.4794 & 0.5665 \end{pmatrix}_{5 \times 3}$$

**Step 7.** Utilize Eqs. (31) and (32) to identify the group positive ideal solution (GPIS) and group negative ideal solution (GNIS), respectively, as follows:

$$X_+^G = \{0.5683, 0.4794, 0.6113\}$$
$$X_-^G = \{0.3964, 0.3735, 0.4902\}$$

**Step 8.** Utilize Eqs. (33) and (34) to calculate the separation measures $d_{+i}^G$ and $d_{-i}^G$ of each alternative $x_i$ from the group positive ideal solution $X_+^G$ and the group negative ideal solution $X_-^G$, respectively, as follows:

$d_{+1}^G = 0.0353$, $d_{-1}^G = 0.0977$, $d_{+2}^G = 0.0621$, $d_{-2}^G = 0.0709$, $d_{+3}^G = 0.0927$, $d_{-3}^G = 0.0402$, $d_{+4}^G = 0.0668$, $d_{-4}^G = 0.0662$, $d_{+5}^G = 0.0722$, $d_{-5}^G = 0.0607$

**Step 9.** Utilize Eq. (35) to calculate the group relative-closeness coefficient $C_i^G$ of each alternative $x_i$ to group positive ideal solution $d_{+i}^G$ as:

$C_1^G = 0.7344$, $C_2^G = 0.5330$, $C_3^G = 0.3027$, $C_4^G = 0.4978$, $C_5^G = 0.4567$

**Step 10:** Rank the alternatives $x_i$ ($i = 1,2,3,4,5$) according to the group relative-closeness coefficient $C_i^G$ ($i = 1,2,3,4,5$). Clearly, $x_1 \succ x_2 \succ x_4 \succ x_5 \succ x_3$, and thus the best alternative is $x_1$.

**Case 2.** The information about the attribute weights is partly known and the known weight information is given as follows:

$$\Delta = \left\{ 0.18 \le w_1 \le 0.2,\ 0.15 \le w_2 \le 0.25,\ 0.3 \le w_3 \le 0.35,\ 0.3 \le w_4 \le 0.4,\ w_j \ge 0,\ j = 1,2,3,4,\ \sum_{j=1}^{4} w_j = 1 \right\}$$

**Step 1'.** See Step 1.

**Step 2'.** See Step 2.

**Step 3'.** Utilize the model (M-2) to construct the single-objective model as follows:

$$\begin{cases} \max D(w) = 4.8000w_1 + 6.0000w_2 + 5.5000w_3 + 4.1333w_4 \\ \text{s.t.} \quad w \in \Delta \end{cases}$$

By solving this model, we get the optimal weight vector of attributes $w = (0.1800, 0.1500, 0.3000, 0.3700)^T$.

**Step 4'.** See Step 4.

**Step 5'.** Utilize Eqs. (27) and (28) to calculate the separation measures $d_{+i}^{(k)}$ and $d_{-i}^{(k)}$ of each alternative $x_i$ of the decision maker $d_k$:

$d_{+1}^{(1)} = 0.2597$, $d_{-1}^{(1)} = 0.2593$, $d_{+2}^{(1)} = 0.2603$, $d_{-2}^{(1)} = 0.2587$, $d_{+3}^{(1)} = 0.2625$, $d_{-3}^{(1)} = 0.2565$, $d_{+4}^{(1)} = 0.2565$, $d_{-4}^{(1)} = 0.2625$, $d_{+5}^{(1)} = 0.3157$, $d_{-5}^{(1)} = 0.2033$, $d_{+1}^{(2)} = 0.2395$, $d_{-1}^{(2)} = 0.2335$, $d_{+2}^{(2)} = 0.2817$, $d_{-2}^{(2)} = 0.1913$, $d_{+3}^{(2)} = 0.2213$, $d_{-3}^{(2)} = 0.2517$,

$d_{+4}^{(2)} = 0.3562$ , $d_{-4}^{(2)} = 0.1168$ , $d_{+5}^{(2)} = 0.2220$ , $d_{-5}^{(2)} = 0.2510$ ,

$d_{+1}^{(3)} = 0.2587$ , $d_{-1}^{(3)} = 0.2160$ , $d_{+2}^{(3)} = 0.1873$ , $d_{-2}^{(3)} = 0.2875$ , $d_{+3}^{(3)} = 0.2400$ , $d_{-3}^{(3)} = 0.2347$ ,

$d_{+4}^{(3)} = 0.1985$ , $d_{-4}^{(3)} = 0.2762$ , $d_{+5}^{(3)} = 0.1628$ , $d_{-5}^{(3)} = 0.3120$ .

**Step 6'.** Utilize Eq. (29) to calculate the relative closeness coefficient $C_i^{(k)}$ of each alternative $x_i$ to the hesitant fuzzy linguistic PIS $X_+^{(k)}$ of the decision maker $d_k$ as

$C_1^{(1)} = 0.4995$ , $C_2^{(1)} = 0.4986$ , $C_3^{(1)} = 0.4942$ , $C_4^{(1)} = 0.5058$ , $C_5^{(1)} = 0.3916$ ,

$C_1^{(2)} = 0.4937$ , $C_2^{(2)} = 0.4043$ , $C_3^{(2)} = 0.5322$ , $C_4^{(2)} = 0.2468$ , $C_5^{(2)} = 0.5307$ ,

$C_1^{(3)} = 0.4550$ , $C_2^{(3)} = 0.6056$ , $C_3^{(3)} = 0.4945$ , $C_4^{(3)} = 0.5819$ , $C_5^{(3)} = 0.6572$ .

Then, we construct the relative-closeness coefficient matrix as below:

$$C = \begin{pmatrix} 0.4995 & 0.4937 & 0.4550 \\ 0.4986 & 0.4043 & 0.6056 \\ 0.4942 & 0.5322 & 0.4945 \\ 0.5058 & 0.2468 & 0.5819 \\ 0.3916 & 0.5307 & 0.6572 \end{pmatrix}_{5 \times 3}$$

**Step 7'.** Utilize Eqs. (31) and (32) to identify the group positive ideal solution (GPIS) and group negative ideal solution (GNIS), respectively as follows:

$$X_+^G = \{0.5058, 0.5322, 0.6572\}$$
$$X_-^G = \{0.3916, 0.2468, 0.4550\}$$

**Step 8'.** Utilize Eqs. (33) and (34) to calculate the separation measures $d_{+i}^G$ and $d_{-i}^G$ of each alternative $x_i$ from the group positive ideal solution $X_+^G$ and the group negative ideal solution $X_-^G$, respectively, as follows:

$d_{+1}^G = 0.0824$ , $d_{-1}^G = 0.1182$ , $d_{+2}^G = 0.0622$ , $d_{-2}^G = 0.1383$ , $d_{+3}^G = 0.0581$ , $d_{-3}^G = 0.1425$ ,

$d_{+4}^G = 0.1202$ , $d_{-4}^G = 0.0804$ , $d_{+5}^G = 0.0386$ , $d_{-5}^G = 0.1620$

**Step 9'.** Utilize Eq. (35) to calculate the group relative-closeness coefficient $C_i^G$ of each alternative $x_i$ to group positive ideal solution $d_{+i}^G$ as:

$C_1^G = 0.5895$ , $C_2^G = 0.6897$ , $C_3^G = 0.7104$ , $C_4^G = 0.4006$ , $C_5^G = 0.8077$

**Step 10.** Rank the alternatives $x_i$ ( $i = 1, 2, 3, 4, 5$ ) according to the group relative-closeness coefficient $C_i^G$ ( $i = 1, 2, 3, 4, 5$ ). Clearly, $x_5 \succ x_3 \succ x_2 \succ x_1 \succ x_4$ , and thus the best alternative is $x_5$ .

# 5. COMPARISON WITH THE OTHER HESITANT FUZZY MULTIPLE ATTRIBUTE DECISION MAKING METHODS

In this section, we will perform a comparison analysis between our new method and the other existing hesitant fuzzy multi-attribute decision making methods, and highlight the advantages of the new method.

## 5.1 Comparison with the Hesitant Fuzzy Multiple Attribute Decision Making (MADM) Methods Based on TOPSIS

Zhang and Wei [54] extended the concept of TOPSIS method to develop a methodology for solving MADM problems with hesitant fuzzy element. Xu and Zhang [55] developed a method based on TOPSIS and the maximizing deviation method for solving MADM problems, in which the attribute values provided by the decision makers are expressed in hesitant fuzzy elements and the information about attribute weights is incomplete. Moreover, they extended the developed method to interval-valued hesitant fuzzy situations. Compared to Zhang and Wei's method and Xu and Zhang's method, the advantages of our method are as follows:

(1) Zhang and Wei's method and Xu and Zhang's method aim at dealing with the MADM problem in which the attribute values take the form of hesitant fuzzy elements, and they can not be used to accommodate the hesitant fuzzy linguistic information. On the contrary, our method can be applied to MAGDM problems in which the attribute values take the form of HFLTSs, and therefore it can be used to address the hesitant fuzzy linguistic information.

(2) Zhang and Wei's method and Xu and Zhang's method focus on the MADM problems. However, in real-life, due to the increasing complexity of socio-economic environment, it is less and less possible for a single decision maker to consider all relevant aspects of the problem. Therefore, many organizations employ groups to make decision, which is called as group decision making (GDM). Our method gives a TOPIS based procedure to solve a MAGDM problem under hesitant fuzzy linguistic environments. First, in our method, a quadratic programming model is established to determine the weights of decision makers, which is not be discussed in Zhang and Wei's method and Xu and Zhang's method. In addition, Zhang and Wei's method doesn't consider the weights of attributes. Xu and Zhang [55] established an optimization model to determine the attribute weights. But, this model determined the attribute weights from only an individual hesitant fuzzy decision matrix, and it cannot determine the importance weights of attributes in group decision making. Our method can derive the optimal weights of attributes from all individual hesitant fuzzy linguistic decision matrices. Finally, the TOPSIS methods proposed by Zhang and Wei [54] and Xu and Zhang [55] only included a stage, which we call the hesitant fuzzy TOPSIS; while the extended TOPSIS proposed by our method includes two stages. The first stage is called the hesitant fuzzy linguistic TOPSIS, which can be used to calculate the individual relative closeness coefficient of each alternative to the individual hesitant fuzzy linguistic PIS. The second stage is the standard TOPSIS, which is used to calculate the group relative-closeness coefficient of each alternative to group PIS and select the optimal one with the maximum group relative-closeness coefficient.

### 5.2 Comparison with the Group Decision Making Model Dealing with Comparative Linguistic Expressions Based on Hesitant Fuzzy Linguistic Term Sets

Rodríguez et al. [31] proposed a group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. In the following, we will compare our method with the Rodríguez et al.'s model to illustrate the advantages of our method. First, we revisit Example 4.1 by using the Rodríguez et al.'s model.

**Step 1.** Obtaining for each HFLTS $A_{ij}^{(k)}$ ($i=1,2,3,4,5$, $j=1,2,3,4$, $k=1,2,3$) its envelope and then get three uncertain linguistic decision matrices $\mathcal{P}^{(k)} = \left( P_{ij}^{(k)} \right)_{5 \times 4}$ ($k=1,2,3$) (see Tables 7-9).

**Table 7. Uncertain linguistic decision matrix $\mathcal{P}^{(1)}$**

| 7 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|------|-----------|-----------|-----------|-----------|
| $x_1$ | $[s_5, s_7]$ | $[s_5, s_8]$ | $[s_3, s_5]$ | $[s_2, s_5]$ |
| $x_2$ | $[s_5, s_6]$ | $[s_4, s_8]$ | $[s_1, s_4]$ | $[s_4, s_6]$ |
| $x_3$ | $[s_2, s_3]$ | $[s_7, s_8]$ | $[s_2, s_4]$ | $[s_5, s_6]$ |
| $x_4$ | $[s_4, s_7]$ | $[s_3, s_5]$ | $[s_7, s_8]$ | $[s_1, s_3]$ |
| $x_5$ | $[s_4, s_8]$ | $[s_2, s_5]$ | $[s_4, s_6]$ | $[s_1, s_4]$ |

**Table 8. Uncertain linguistic decision matrix $\mathcal{P}^{(2)}$**

| 8 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|------|-----------|-----------|-----------|-----------|
| $x_1$ | $[s_5, s_7]$ | $[s_1, s_4]$ | $[s_4, s_8]$ | $[s_2, s_5]$ |
| $x_2$ | $[s_3, s_7]$ | $[s_5, s_6]$ | $[s_5, s_8]$ | $[s_1, s_2]$ |
| $x_3$ | $[s_7, s_8]$ | $[s_2, s_4]$ | $[s_3, s_5]$ | $[s_4, s_6]$ |
| $x_4$ | $[s_4, s_7]$ | $[s_7, s_8]$ | $[s_2, s_4]$ | $[s_1, s_3]$ |
| $x_5$ | $[s_5, s_7]$ | $[s_2, s_5]$ | $[s_7, s_8]$ | $[s_1, s_4]$ |

**Table 9. Uncertain linguistic decision matrix $\mathcal{P}^{(3)}$**

| 9 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $[s_5,s_8]$ | $[s_6,s_7]$ | $[s_3,s_5]$ | $[s_2,s_5]$ |
| $x_2$ | $[s_4,s_6]$ | $[s_1,s_4]$ | $[s_4,s_8]$ | $[s_5,s_6]$ |
| $x_3$ | $[s_2,s_3]$ | $[s_6,s_8]$ | $[s_2,s_4]$ | $[s_5,s_8]$ |
| $x_4$ | $[s_1,s_3]$ | $[s_5,s_6]$ | $[s_4,s_8]$ | $[s_4,s_7]$ |
| $x_5$ | $[s_7,s_8]$ | $[s_1,s_4]$ | $[s_4,s_6]$ | $[s_5,s_7]$ |

**Step 2.** Utilize the arithmetic mean aggregation operator based on 2-tuple [56] to obtain the pessimistic and optimistic collective 2-tuple linguistic decision matrices $\mathcal{P}^- = \left(P_{ij}^-\right)_{5\times4}$ and $\mathcal{P}^+ = \left(P_{ij}^+\right)_{5\times4}$ (see Tables 10 and 11), respectively. Since the Rodríguez et al.'s model needs to know the weight values of decision makers in advance, hence we assume the weight vector of decision makers as $\omega = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

For example,

$$P_{12}^- = \Delta\left(\frac{1}{3}\Delta^{-1}(s_5,0)+\frac{1}{3}\Delta^{-1}(s_1,0)+\frac{1}{3}\Delta^{-1}(s_6,0)\right)=(s_4,0)$$

$$P_{12}^+ = \Delta\left(\frac{1}{3}\Delta^{-1}(s_8,0)+\frac{1}{3}\Delta^{-1}(s_4,0)+\frac{1}{3}\Delta^{-1}(s_7,0)\right)=\left(s_6,\frac{1}{3}\right)$$

**Step 3.** Computing a pessimistic and optimistic collective overall preference values $P_i^-$ and $P_i^+$ ($i=1,2,3,4,5$) for each alternative (see Table 12). Since the Rodríguez et al.'s model needs to know the weight values of attributes in advance, hence we assume the weight vector of attributes as $w = \left(0.2349,0.2936,0.2692,0.2023\right)^T$ .

**Table 10. Pessimistic collective 2-tuple linguistic decision matrix $\mathcal{P}^-$**

| 10 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $(s_5,0)$ | $(s_4,0)$ | $\left(s_3,\frac{1}{3}\right)$ | $(s_2,0)$ |
| $x_2$ | $(s_4,0)$ | $\left(s_3,\frac{1}{3}\right)$ | $\left(s_3,\frac{1}{3}\right)$ | $\left(s_3,\frac{1}{3}\right)$ |
| $x_3$ | $\left(s_4,-\frac{1}{3}\right)$ | $(s_5,0)$ | $\left(s_2,\frac{1}{3}\right)$ | $\left(s_5,-\frac{1}{3}\right)$ |
| $x_4$ | $(s_3,0)$ | $(s_5,0)$ | $\left(s_4,\frac{1}{3}\right)$ | $(s_2,0)$ |
| $x_5$ | $\left(s_5,\frac{1}{3}\right)$ | $\left(s_2,-\frac{1}{3}\right)$ | $(s_5,0)$ | $\left(s_2,\frac{1}{3}\right)$ |

**Table 11. Optimistic collective 2-tuple linguistic decision matrix $\mathcal{P}^+$**

| 11 | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $\left(s_7,\frac{1}{3}\right)$ | $\left(s_6,\frac{1}{3}\right)$ | $\left(s_6,0\right)$ | $\left(s_5,0\right)$ |
| $x_2$ | $\left(s_6,\frac{1}{3}\right)$ | $\left(s_6,0\right)$ | $\left(s_7,-\frac{1}{3}\right)$ | $\left(s_5,-\frac{1}{3}\right)$ |
| $x_3$ | $\left(s_5,-\frac{1}{3}\right)$ | $\left(s_7,-\frac{1}{3}\right)$ | $\left(s_4,\frac{1}{3}\right)$ | $\left(s_7,-\frac{1}{3}\right)$ |
| $x_4$ | $\left(s_6,-\frac{1}{3}\right)$ | $\left(s_6,\frac{1}{3}\right)$ | $\left(s_7,-\frac{1}{3}\right)$ | $\left(s_4,\frac{1}{3}\right)$ |
| $x_5$ | $\left(s_8,-\frac{1}{3}\right)$ | $\left(s_5,-\frac{1}{3}\right)$ | $\left(s_7,-\frac{1}{3}\right)$ | $\left(s_5,0\right)$ |

**Table 12. Pessimistic and optimistic collective overall preference values for each alternative**

| 12 | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| pessimistic | $(s_4,-0.3491)$ | $(s_3,0.4899)$ | $(s_4,-0.0984)$ | $(s_4,-0.2561)$ | $(s_4,-0.4399)$ |
| optimistic | $(s_6,0.2088)$ | $(s_6,-0.0120)$ | $(s_6,-0.4312)$ | $(s_6,-0.1381)$ | $(s_6,-0.0228)$ |

For example,

$$P_1^- = \Delta\left(\frac{1}{3}\Delta^{-1}(s_5,0)+\frac{1}{3}\Delta^{-1}(s_4,0)+\frac{1}{3}\Delta^{-1}\left(s_3,\frac{1}{3}\right)+\frac{1}{3}\Delta^{-1}(s_2,0)\right)=(s_4,-0.3491)$$

$$P_1^+ = \Delta\left(\frac{1}{3}\Delta^{-1}\left(s_7,\frac{1}{3}\right)+\frac{1}{3}\Delta^{-1}\left(s_6,\frac{1}{3}\right)+\frac{1}{3}\Delta^{-1}(s_6,0)+\frac{1}{3}\Delta^{-1}(s_5,0)\right)=(s_6,0.2088)$$

**Step 4.** Building a vector of intervals $\mathcal{P}=\left(P_1,P_2,P_3,P_4,P_5\right)$ for the alternatives (see Table 13).

**Table 13. Linguistic intervals for each alternative**

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|
| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
| $\left[\begin{matrix}(s_4,-0.3491),\\(s_6,0.2088)\end{matrix}\right]$ | $\left[\begin{matrix}(s_3,0.4899),\\(s_6,-0.0120)\end{matrix}\right]$ | $\left[\begin{matrix}(s_4,-0.0984),\\(s_6,-0.4312)\end{matrix}\right]$ | $\left[\begin{matrix}(s_4,-0.2561),\\(s_6,-0.1381)\end{matrix}\right]$ | $\left[\begin{matrix}(s_4,-0.4399),\\(s_6,-0.0228)\end{matrix}\right]$ |

**Step 5.** Building a preference relation $\mathcal{V}=\left(V_{ij}\right)_{5\times5}$ (see Table 14).

**Table 14. Preference relation** $\mathcal{V} = \left(V_{ij}\right)_{5\times5}$

| **14** | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $x_1$ | 0.5 | 0.5378 | 0.5461 | 0.5272 | 0.5324 |
| $x_2$ | 0.4622 | 0.5 | 0.5009 | 0.4862 | 0.4940 |
| $x_3$ | 0.4539 | 0.4991 | 0.5 | 0.4821 | 0.4918 |
| $x_4$ | 0.4728 | 0.5138 | 0.5179 | 0.5 | 0.5076 |
| $x_5$ | 0.4676 | 0.5060 | 0.5082 | 0.4924 | 0.5 |

For example,

$$V_{12} = P\left(P_1 > P_2\right) = \frac{\max\left(0, 6.2088 - 3.4899\right) - \max\left(0, 3.6509 - 5.9880\right)}{\left(6.2088 - 3.6509\right) + \left(5.9880 - 3.4899\right)}$$

where

$$P_1 = \left[\Delta^{-1}\left(s_4, -0.3491\right), \Delta^{-1}\left(s_6, 0.2088\right)\right] = \left[3.6509, 6.2088\right]$$

$$P_2 = \left[\Delta^{-1}\left(s_3, 0.4899\right), \Delta^{-1}\left(s_6, -0.0120\right)\right] = \left[3.4899, 5.9880\right]$$

**Step 6.** Calculate the non-dominance degrees $NDD_i$ ( $i = 1, 2, 3, 4, 5$ ) for each alternative. $NDD_1 = 1.0000$ , $NDD_2 = 0.9245$ , $NDD_3 = 0.9079$ , $NDD_4 = 0.9457$ , $NDD_5 = 0.9352$ . For example,

$$NDD_1 = \min\left\{\begin{array}{l} 1 - \max\left(\left(0.4622\text{-}0.5378\right), 0\right), 1 - \max\left(\left(0.4539\text{-}0.5461\right), 0\right), \\ 1 - \max\left(\left(0.4728\text{-}0.5272\right), 0\right), 1 - \max\left(\left(0.4676\text{-}0.5324\right), 0\right) \end{array}\right\} = 1$$

Finally, the set of alternatives is ordered according to $NDD_i$ ( $i = 1, 2, 3, 4, 5$ ) as follows:

$$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$$

And the best alternative is $x_1$ .

It is easy to see that the optimal alternative obtained by the Rodríguez et al.'s method is the same as our method, which shows the effectiveness and reasonableness of our method. However, it is noticed that the ranking order of the alternatives obtained by our method is $x_1 \succ x_2 \succ x_4 \succ x_5 \succ x_3$ , which is different from the ranking order obtained by the Rodríguez et al.'s method. Concretely, the ranking orders between $x_2$ and $x_4$, and between $x_2$ and $x_5$ obtained by the two methods are just converse, i.e., $x_2 \succ x_4$ and $x_2 \succ x_5$ for our method while $x_4 \succ x_2$ and $x_5 \succ x_2$ for the Rodríguez et al.'s method. It is easy to see that our method has some desired advantages over the Rodríguez's method, which are illustrated as follows:

(1) From Step 1 above, we can see that the Rodríguez et al.'s method first transform the HFLTSs to uncertain linguistic variables [35,57,58] and then deal with the transformed uncertain linguistic variables. It is noted that such a transformation pays more attentions to the maximum and minimum linguistic terms in a HFLTS, and neglects the importance of the linguistic terms between the maximum linguistic term and the minimum linguistic term. As a result, it leads to the loss of information, which affects the final ranking results. However, our method does not need to perform such a transformation but directly deals with the HFLTSs, thereby provides a more sufficient and comprehensive description of the differences opinions of the DMs than the Rodríguez et al.'s method. The comparison shows that our method has its great superiority in handling the ambiguity and hesitancy inherent in MAGDM problems with hesitant fuzzy linguistic information.

(2) Our method utilizes the maximizing group consensus method and the maximizing deviation method to determine the weight values of decision makers and attributes, respectively, which is more objective and reasonable; while the Rodríguez et al.'s method asks the DMs to provide the weight values of decision makers and attributes in advance, which is subjective and sometime cannot yield the persuasive results.

## 6. CONCLUSIONS

In the current paper, we have proposed a novel method for hesitant fuzzy linguistic MAGDM with incomplete weight information, which involves three parts:

(1) Based on the idea that a set of group members should have a maximum degree of agreement solution, by maximizing the group consensus, we have first developed a method to determine the optimal weights of decision makers under hesitant fuzzy linguistic situations, which ensures the rationality of the individual decision information.

(2) Then, motivated by the idea that a larger weight should be assigned to the attribute with a larger deviation value among alternatives, we have further proposed a maximizing deviation measure based method to determine the optimal attribute weights under hesitant fuzzy linguistic environments, which eliminates the influence of subjectivity of attribute weights provided by the decision makers in advance.

(3) Moreover, we have proposed an extended TOPSIS method to solve MAGDM problems with hesitant fuzzy linguistic information, which includes the hesitant fuzzy linguistic TOPSIS and the standard TOPSIS. The former is to calculate the relative closeness coefficient of each alternative to the hesitant fuzzy linguistic PIS; while the latter is to calculate the group relative-closeness coefficient of each alternative to group PIS, based on which we rank the considered alternatives and then select the optimal one with the maximum group relative-closeness coefficient. An important advantage of the extended TOPSIS method is that it can avoid the loss of hesitant fuzzy linguistic information in the process of information aggregation.

Finally, an investment example has been used to illustrate the effectiveness and practicality of the developed method. A comparison analysis has also been made to show the advantages of the developed method over the other methods.

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## COMPETING INTERESTS

Author has declared that no competing interests exist.

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