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Conservative Interaction of *N* Internal Waves in Three Dimensions

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Abstract

The Navier-Stokes system of equations is reduced to a system of the vorticity, continuity, Helmholtz, and Lamb-Helmholtz equations. The periodic Dirichlet problems are formulated for internal waves vanishing at infinity in the upper and lower domains. Stationary kinematic Fourier (SKF) structures, stationary exponential kinematic Fourier (SKEF) structures, stationary dynamic exponential (SDEF) Fourier structures, and SKEF-SDEF structures of three spatial variables and time are constructed in the current paper to treat kinematic and dynamic problems of the three-dimensional theory of the Newtonian flows with harmonic velocity. Two exact solutions for conservative interaction of *N* internal waves in three dimensions are developed by the method of decomposition in invariant structures and implemented through experimental and theoretical programming in Maple™. Main results are summarized in a global existence theorem for the strong solutions. The SKEF, SDEF, and SKEF-SDEF structures of the cumulative flows are visualized by two-parametric surface plots for six fluid-dynamic variables.

Keywords

Existence Theorem, Internal Waves, Invariant Structures, Experimental Programming, Theoretical Programming

1. Introduction

The three-dimensional (3d) Navier-Stokes system of partial differential equations (PDEs) for a Newtonian fluid with a constant density ρ and a constant kinematic viscosity ν in a gravity field $\boldsymbol{g}$ is

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\frac{1}{\rho}\nabla p_t+\nu\Delta\boldsymbol{v}+\boldsymbol{g}, \tag{1}$$

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$$\nabla \cdot \boldsymbol{v} = 0, \tag{2}$$

where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ are the gradient and the Laplacian in the Cartesian coordinate system $\boldsymbol{x} = (x, y, z)$ of the 3d space with unit vectors $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$; t is time; $\boldsymbol{v} = (u, v, w)$ is a vector field of the flow velocity; $\boldsymbol{g} = (0, 0, -g_z)$ is a vector field of the gravitational acceleration; p_t is a scalar field of the total pressure.

By a flow vorticity $\boldsymbol{\omega} = (\delta, \kappa, \omega)$ of the velocity field

$$\nabla \times \boldsymbol{v} = \boldsymbol{\omega}, \tag{3}$$

Equation (1) may be written into the Lamb-Pozrikidis form [1] [2]

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\left(\frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + \frac{p_t}{\rho} - \boldsymbol{g}\cdot\boldsymbol{x}\right) + \boldsymbol{\omega}\times\boldsymbol{v} + \nu\nabla\times\boldsymbol{\omega} = \boldsymbol{0}, \tag{4}$$

which sets a dynamic balance of inertial, potential, vortical, and viscous forces, respectively.

Using a dynamic pressure per unit mass [3]

$$p_d = \frac{p_t - p_0}{\rho} - \boldsymbol{g}\cdot\boldsymbol{x}, \tag{5}$$

where p_0 is a reference pressure, a kinetic energy per unit mass $k_e = \boldsymbol{v}\cdot\boldsymbol{v}/2$, the 3d Helmholtz decomposition [4] of the velocity field

$$\boldsymbol{v} = \nabla\phi + \nabla\times\boldsymbol{\psi}, \tag{6}$$

$$\nabla\cdot\boldsymbol{\psi} = \boldsymbol{0}, \tag{7}$$

and the vortex force

$$\boldsymbol{\omega}\times\boldsymbol{v} = \nabla d + \nabla\times\boldsymbol{a}, \tag{8}$$

$$\nabla\cdot\boldsymbol{a} = \boldsymbol{0}, \tag{9}$$

Equation (4) is reduced to the Lamb-Helmholtz PDE [5]

$$\nabla b_e + \nabla\times\boldsymbol{h}_e = \boldsymbol{0} \tag{10}$$

for a scalar Bernoulli potential and a vector Helmholtz potential $\boldsymbol{h}_e = (fe, ge, he)$, respectively,

$$b_e = \frac{\partial\phi}{\partial t} + p_d + k_e + d, \tag{11}$$

$$\boldsymbol{h}_e = \frac{\partial\boldsymbol{\psi}}{\partial t} + \nu\boldsymbol{\omega} + \boldsymbol{a}, \tag{12}$$

where ϕ, d are scalar potentials and $\boldsymbol{\psi} = (\chi, \eta, \psi)$, $\boldsymbol{a} = (a, b, c)$ are vector potentials of $\boldsymbol{v}$ and $\boldsymbol{\omega}\times\boldsymbol{v}$, respectively. The Lamb-Helmholtz PDE (10) sets a dynamic balance between potential and vortical forces of the Navier-Stokes PDE (1), which are separated completely. Reduction of (1) to (10) means the potential-vortical duality of the Navier-Stokes PDE for free flows [3] since writing Equation (10) as

$$\boldsymbol{n}_s = -\nabla b_e = \nabla\times\boldsymbol{h}_e \tag{13}$$

shows that a virtual force $\boldsymbol{n}_s$ of (1) may be represented both in the potential form $\boldsymbol{n}_s = -\nabla b_e$ and the vortical form $\boldsymbol{n}_s = \nabla\times\boldsymbol{h}_e$. For instance, the potential-vortical duality of (1)-(2) results in formation of the wave-vortex structures in surface waves [6]-[8].

The exponential Fourier eigenfunctions were calculated by separation of variables of the 3d Laplace equation, for instance, see [1] and [4], and applied for a linear part of the kinematic problem for free-surface waves in the theory of the ideal fluid with $\nu = 0$ in [9]. The analytical method of separation of variables was recently generalized into the computational method of solving PDEs by decomposition into invariant structures. The Boussinesq-Rayleigh-Taylor structures were used to compute topological flows away from boundaries in [3]. The trigonometric Taylor structures and the trigonometric-hyperbolic structures were developed in [10] to model spatiotemporal cascades of exposed and hidden perturbations of the Couette flow. In [11], the invariant trigonometric, hyperbolic, and elliptic structures were utilized to treat dual perturbations of the Poiseuille-Hagen flow. The zigzag hyperbolic structures were studied to derive the exact solution for interaction of two pulsatory waves of the Korteweg-de Vries equation in [12]. In two dimensions, the stationary kinematic Fourier (SKF) structures with space-dependent structural coefficients, the stationary exponential kinematic Fourier (SKEF) structures, the stationary dynamic exponential Fourier (SDEF) structures, and SKEF-SDEF structures with constant structural coefficients were developed to obtain the exact solutions of the Navier-Stokes system of PDEs for conservative interaction of N internal waves by the experimental and theoretical programming [5].

In the current paper, the SKF, SKEF, SDEF, and SKEF-SDEF structures are extended in three dimensions to examine kinematic and dynamic problems for internal conservative waves in the theory of Newtonian flows with harmonic velocity. The structure of this paper is as follows. The SKF structures are used to compute theoretical solutions for the velocity components in Section 2. Theoretical solutions for the kinematic potentials of the velocity field and the dynamic potentials of the Navier-Stokes PDE are obtained in Sections 3 and 4, respectively, through the SKEF structures. The SDEF structures are constructed in Section 5. In Section 6, the SDEF and SKEF-SDEF structures are used for theoretical computation of the kinetic energy and the dynamic pressure. Decomposition of harmonic variables in a SKEF structural basis is tackled in Section 7. Verification of the experimental and theoretical solutions by the Navier-Stokes system of PDEs and the existence theorem are provided in Section 8. Discussion of significant outcomes and visualization of the developed structures are given in Section 9, which is followed by a summary of main results in Section 10.

2. Velocity Components in the SKF Structures

The following theoretical solutions and admissible boundary conditions of Sections 2-8 were primarily computed in Maple™ using experimental programming with lists of equations and expressions for numerical indices and $N = 3$ in the virtual environment of a global variable *Eqe* by 33 developed procedures of 1748 code lines.

Theoretical problems for harmonic velocity components $u = u(x, y, z, t)$, $v = v(x, y, z, t)$, $w = w(x, y, z, t)$ of cumulative flows $\boldsymbol{v} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$ of a Newtonian fluid are given by vanishing the x-, y-, z- components of the vorticity Equation (3) and the continuity Equation (2), respectively,

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \tag{14}$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \tag{15}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \tag{16}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{17}$$

To consider conservative interaction of N internal waves, the cumulative flows are decomposed into superpositions of local flows

$$u = \sum_{n=1}^{N} u_n(x, y, z, t), \quad v = \sum_{n=1}^{N} v_n(x, y, z, t), \quad w = \sum_{n=1}^{N} w_n(x, y, z, t), \tag{18}$$

such that the local vorticity and continuity equations are

$$\frac{\partial w_n}{\partial y} - \frac{\partial v_n}{\partial z} = 0, \tag{19}$$

$$\frac{\partial u_n}{\partial z} - \frac{\partial w_n}{\partial x} = 0, \tag{20}$$

$$\frac{\partial v_n}{\partial x} - \frac{\partial u_n}{\partial y} = 0, \tag{21}$$

$$\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y} + \frac{\partial w_n}{\partial z} = 0, \tag{22}$$

where $n = 1, 2, \cdots, N$.

An upper cumulative flow is specified by a Dirichlet condition, which is periodic in the *x*- and *y*-directions, through the two-dimensional (2d) SKF structure on a lower boundary $z = 0$ of an upper domain $x \in (-\infty, \infty)$, $y \in (-\infty, \infty)$, $z \in [0, \infty)$ (see **Figure 1**)

$$w|_{z=0} = \sum_{n=1}^{N} \left(Fw_n cc_n + Qw_n cs_n + Gw_n sc_n + Rw_n ss_n \right), \tag{23}$$

and a vanishing Dirichlet condition in the *z*-direction

$$w|_{z=\infty} = 0. \tag{24}$$

A lower cumulative flow is identified by a periodic Dirichlet condition on an upper boundary $z = 0$ of a lower domain $x \in (-\infty, \infty)$, $y \in (-\infty, \infty)$, $z \in (-\infty, 0]$ (see **Figure 1**)

$$w|_{z=0} = \sum_{n=1}^{N} \left(Fw_n cc_n + Qw_n cs_n + Gw_n sc_n + Rw_n ss_n \right), \tag{25}$$

and a vanishing Dirichlet condition in the *z*-direction

$$w|_{z=-\infty} = 0. \tag{26}$$

Thus, an effect of surface waves on the internal waves is described by the Dirichlet conditions (23) and (25). While notations of boundary coefficients Fw_n, Qw_n, Gw_n, Rw_n coincide in (23) and (25) for computational simplicity, their values are different for the upper and lower flows, which model internal waves produced by surface waves in atmosphere and ocean. In Equations (23) and (25), a structural notation

$$cc_n = \cos(\alpha_n)\cos(\beta_n), \quad cs_n = \cos(\alpha_n)\sin(\beta_n), \quad sc_n = \sin(\alpha_n)\cos(\beta_n), \quad ss_n = \sin(\alpha_n)\sin(\beta_n), \tag{27}$$

is used for kinematic structural functions cc_n, cs_n, sc_n, ss_n, where $\alpha_n = \rho_n X_n$, $\beta_n = \sigma_n Y_n$ are arguments of the kinematic and dynamic structural functions, $X_n = x - Cx_n t - Xa_n$, $Y_n = y - Cy_n t - Yb_n$ are propagation variables, ρ_n, σ_n are wave numbers, Cx_n, Cy_n are celerities, and Xa_n, Yb_n are initial coordinates for all *n*.

The experimental solutions show that similar to [5], boundary conditions for u_n, v_n are then redundant since boundary parameters of u_n, v_n depend on boundary parameters of w_n for the upper and lower flows, respectively, as

$$\begin{aligned} u|_{z=0} &= \mp \sum_{n=1}^{N} \frac{\rho_n}{\sqrt{\rho_n^2 + \sigma_n^2}} \left(Gw_n cc_n + Rw_n cs_n - Fw_n sc_n - Qw_n ss_n \right), \\ v|_{z=0} &= \mp \sum_{n=1}^{N} \frac{\sigma_n}{\sqrt{\rho_n^2 + \sigma_n^2}} \left(Qw_n cc_n - Fw_n cs_n + Rw_n sc_n - Gw_n ss_n \right), \end{aligned} \tag{28}$$

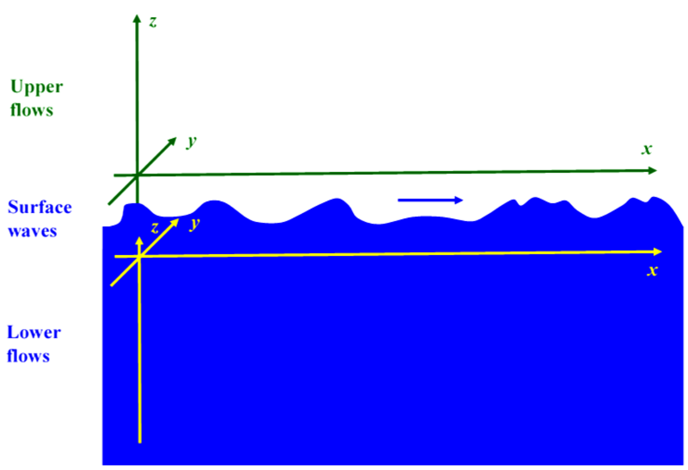

Figure 1. Configuration of upper and lower domains for conservative waves.

Similarly to *w*, *u* and *v* vanish as $z \to \pm\infty$

$$u\big|_{z=\pm\infty} = 0, \quad v\big|_{z=\pm\infty} = 0, \tag{29}$$

for the upper and lower cumulative flows, respectively.

Theoretical solutions of (14)-(26) are constructed in the SKF structure $p = p(x, y, z, t)$ of three spatial variables *x*, *y*, *z* and time *t* with a general term p_n, which in the structural notation may be written as

$$p = \sum_{n=1}^{N} p_n(x, y, z, t) = \sum_{n=1}^{N} \left[fp_n(z) cc_n + qp_n(z) cs_n + gp_n(z) sc_n + rp_n(z) ss_n \right], \tag{30}$$

where first letters *f*, *q*, *g*, *r* of space-dependent structural coefficients $fp_n(z)$, $qp_n(z)$, $gp_n(z)$, $rp_n(z)$ refer to the kinematic structural functions cc_n, cs_n, sc_n, ss_n and a second letter to the expanded variable *p*. General terms of the velocity components of the local flows in the structural notation become

$$\begin{aligned} u_n &= fu_n(z) cc_n + qu_n(z) cs_n + gu_n(z) sc_n + ru_n(z) ss_n, \\ v_n &= fv_n(z) cc_n + qv_n(z) cs_n + gv_n(z) sc_n + rv_n(z) ss_n, \\ w_n &= fw_n(z) cc_n + qw_n(z) cs_n + gw_n(z) sc_n + rw_n(z) ss_n. \end{aligned} \tag{31}$$

Computation of spatial derivatives of p_n gives

$$\frac{\partial p_n}{\partial x} = \rho_n \left[gp_n(z) cc_n + rp_n(z) cs_n - fp_n(z) sc_n - qp_n(z) ss_n \right], \tag{32}$$

$$\frac{\partial p_n}{\partial y} = \sigma_n \left[qp_n(z) cc_n - fp_n(z) cs_n + rp_n(z) sc_n - gp_n(z) ss_n \right], \tag{33}$$

$$\frac{\partial p_n}{\partial z} = \frac{\mathrm{d}fp_n}{\mathrm{d}z} cc_n + \frac{\mathrm{d}qp_n}{\mathrm{d}z} cs_n + \frac{\mathrm{d}gp_n}{\mathrm{d}z} sc_n + \frac{\mathrm{d}rp_n}{\mathrm{d}z} ss_n. \tag{34}$$

Application of (32)-(34) to (31) and substitution in (19)-(22) reduce the four PDEs to three ordinary differential equations (ODEs) and an algebraic equation (AE). For these equations to be satisfied exactly for all independent variables, independent parameters, structural functions, and structural coefficients of the local flows *x*, *y*, *z*, *t*, Xa_n, Yb_n, Cx_n, Cy_n, ρ_n, σ_n, cc_n, cs_n, sc_n, ss_n, fu_n, qu_n, gu_n, ru_n, fv_n, qv_n, gv_n, rv_n, fw_n, qw_n, gw_n, rw_n, all coefficients of the kinematic structural functions must vanish. Vanishing four coefficients of four equations yields 16 equations, which are separated into four systems of four equations each with respect to four groups of the SKF structural coefficients of the velocity components $\left[fu_n(z), rv_n(z), gw_n(z) \right]$, $\left[qu_n(z), gv_n(z), rw_n(z) \right]$, $\left[gu_n(z), qv_n(z), fw_n(z) \right]$, $\left[ru_n(z), fv_n(z), qw_n(z) \right]$.

$$-\frac{\mathrm{d}rv_n}{\mathrm{d}z} - \sigma_n gw_n = 0, \quad \frac{\mathrm{d}fu_n}{\mathrm{d}z} - \rho_n gw_n = 0, \quad \sigma_n fu_n + \rho_n rv_n = 0, \quad -\rho_n fu_n + \sigma_n rv_n + \frac{\mathrm{d}gw_n}{\mathrm{d}z} = 0, \tag{35}$$

$$-\frac{\mathrm{d}gv_n}{\mathrm{d}z} + \sigma_n rw_n = 0, \quad \frac{\mathrm{d}qu_n}{\mathrm{d}z} - \rho_n rw_n = 0, \quad -\sigma_n qu_n + \rho_n gv_n = 0, \quad -\rho_n qu_n - \sigma_n gv_n + \frac{\mathrm{d}rw_n}{\mathrm{d}z} = 0, \tag{36}$$

$$-\frac{\mathrm{d}qv_n}{\mathrm{d}z} - \sigma_n fw_n = 0, \quad \frac{\mathrm{d}gu_n}{\mathrm{d}z} + \rho_n fw_n = 0, \quad \sigma_n gu_n - \rho_n qv_n = 0, \quad \rho_n gu_n + \sigma_n qv_n + \frac{\mathrm{d}fw_n}{\mathrm{d}z} = 0, \tag{37}$$

$$-\frac{\mathrm{d}fv_n}{\mathrm{d}z} + \sigma_n qw_n = 0, \quad \frac{\mathrm{d}ru_n}{\mathrm{d}z} + \rho_n qw_n = 0, \quad -\sigma_n ru_n - \rho_n fv_n = 0, \quad \rho_n ru_n - \sigma_n fv_n + \frac{\mathrm{d}qw_n}{\mathrm{d}z} = 0. \tag{38}$$

In these four separated systems, first and second equations, which are produced by (19) and (20), are ODEs; third equations, which are generated by (21), are AEs; and fourth equations, which are created by (22), are again ODEs.

Solving the third AEs of separated systems (35)-(38) yields functional relations between structural coefficients:

$$fu_n = -\frac{\rho_n}{\sigma_n} rv_n, \quad qu_n = \frac{\rho_n}{\sigma_n} gv_n, \quad gu_n = \frac{\rho_n}{\sigma_n} qv_n, \quad ru_n = -\frac{\rho_n}{\sigma_n} fv_n. \tag{39}$$

Substitution of (39) in the second ODEs and addition/subtraction of the first ODEs reduces the second ODEs to identities. Substitution of functional relations (39) into the fourth ODEs reduces them to the following system:

$$\begin{aligned}&\left(\rho_n^2+\sigma_n^2\right)rv_v+\sigma_n\frac{\mathrm{d}gw_n}{\mathrm{d}z}=0,\quad -\left(\rho_n^2+\sigma_n^2\right)gv_v+\sigma_n\frac{\mathrm{d}rw_n}{\mathrm{d}z}=0,\\&\left(\rho_n^2+\sigma_n^2\right)qv_v+\sigma_n\frac{\mathrm{d}fw_n}{\mathrm{d}z}=0,\quad -\left(\rho_n^2+\sigma_n^2\right)fv_v+\sigma_n\frac{\mathrm{d}qw_n}{\mathrm{d}z}=0.\end{aligned}\tag{40}$$

Construct solutions of the first and fourth ODEs in stationary exponential (SE) structures with the following general terms:

$$\left(fv_n,qv_n,gv_n,rv_n,fw_n,qw_n,gw_n,rw_n\right)(z)=\left(Fv_n,Qv_n,Gv_n,Rv_n,Fw_n,Qw_n,Gw_n,Rw_n\right)\exp\left(c_n z\right),\tag{41}$$

where Fv_n, Qv_n, Gv_n, Rv_n, Fw_n, Qw_n, Gw_n, Rw_n, and c_n are structural coefficients. Substitution of (41) in first ODEs of (35)-(38) yields algebraic relations between parameters of the structural coefficients

$$Fv_n=\frac{\sigma_n}{c_n}Qw_n,\quad Qv_n=-\frac{\sigma_n}{c_n}Fw_n,\quad Gv_n=\frac{\sigma_n}{c_n}Rw_n,\quad Rv_n=-\frac{\sigma_n}{c_n}Gw_n.\tag{42}$$

Substitution of (41) into (40) returns admissible values of c_n for the upper and lower flows, respectively, as

$$c_n=-r_n=-\sqrt{\rho_n^2+\sigma_n^2},\quad c_n=r_n=\sqrt{\rho_n^2+\sigma_n^2}.\tag{43}$$

Finally, substitutions of (39) and (41)-(43) into (31) and (18) give the following velocity components of the upper and lower cumulative flows, respectively:

$$\begin{aligned}u&=\mp\sum_{n=1}^{N}\frac{\rho_n}{r_n}\left(Gw_n cc_n+Rw_n cs_n-Fw_n sc_n-Qw_n ss_n\right)\exp\left(\mp r_n z\right),\\v&=\mp\sum_{n=1}^{N}\frac{\sigma_n}{r_n}\left(Qw_n cc_n-Fw_n cs_n+Rw_n sc_n-Gw_n ss_n\right)\exp\left(\mp r_n z\right),\\w&=\sum_{n=1}^{N}\left(Fw_n cc_n+Qw_n cs_n+Gw_n sc_n+Rw_n ss_n\right)\exp\left(\mp r_n z\right).\end{aligned}\tag{44}$$

Thus, the velocity components are resolved through the 3d SKEF structures. If $\sigma_n=0$, then $cs_n=0$, $ss_n=0$, and the 3d solution (44) is reduced to the 2d solution in the x-z plane [5]

$$u=\mp\sum_{n=1}^{N}\left(Gw_n ca_n-Fw_n sa_n\right)\exp\left(\mp\rho_n z\right),\quad v=0,\quad w=\sum_{n=1}^{N}\left(Fw_n ca_n+Gw_n sa_n\right)\exp\left(\mp\rho_n z\right),\tag{45}$$

where $ca_n=\cos\left(\alpha_n\right)$, $sa_n=\sin\left(\alpha_n\right)$. If $\rho_n=0$, then $sc_n=0$, $ss_n=0$, and the 3d solution (44) is transformed into a 2d solution in the y-z plane

$$u=0,\quad v=\mp\sum_{n=1}^{N}\left(Qw_n cb_n-Fw_n sb_n\right)\exp\left(\mp\sigma_n z\right),\quad w=\sum_{n=1}^{N}\left(Fw_n cb_n+Qw_n sb_n\right)\exp\left(\mp\sigma_n z\right),\tag{46}$$

where $cb_n=\cos\left(\beta_n\right)$, $sb_n=\sin\left(\beta_n\right)$.

Therefore, structural parameters of (44)

$$\frac{\rho_n}{r_n}=\frac{\rho_n}{\sqrt{\rho_n^2+\sigma_n^2}}=\cos\left(\theta_n\right)=C_n,\quad \frac{\sigma_n}{r_n}=\frac{\sigma_n}{\sqrt{\rho_n^2+\sigma_n^2}}=\sin\left(\theta_n\right)=S_n\tag{47}$$

return cosine and sine of local front angles θ_n with respect to x-axis. In the general case, the local front angles θ_n differ from local celerity angles φ_n, which are defined by

$$\frac{Cx_n}{\sqrt{Cx_n^2+Cy_n^2}}=\cos\left(\varphi_n\right),\quad \frac{Cy_n}{\sqrt{Cx_n^2+Cy_n^2}}=\sin\left(\varphi_n\right).\tag{48}$$

In the case of resonance propagation with $\rho_n/\sigma_n=Cx_n/Cy_n$, $\theta_n=\varphi_n$ similar to the case of 2d internal waves.

3. Kinematic Potentials through the SKEF Structures

Theoretical problems for the kinematic potentials $\boldsymbol{\psi}=(\chi,\eta,\psi)$ and ϕ of $\boldsymbol{v}$ are set by seven global Helmholtz PDEs (6)-(7)

$$\frac{\partial\psi}{\partial y}-\frac{\partial\eta}{\partial z}-u=0, \tag{49}$$

$$\frac{\partial\chi}{\partial z}-\frac{\partial\psi}{\partial x}-v=0, \tag{50}$$

$$\frac{\partial\eta}{\partial x}-\frac{\partial\chi}{\partial y}-w=0, \tag{51}$$

$$\frac{\partial\chi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\psi}{\partial z}=0, \tag{52}$$

$$\frac{\partial\phi}{\partial x}-u=0, \tag{53}$$

$$\frac{\partial\phi}{\partial y}-v=0, \tag{54}$$

$$\frac{\partial\phi}{\partial z}-w=0, \tag{55}$$

since a scalar-vector duality the velocity field admits two descriptions: a scalar description $\boldsymbol{v}=\nabla\phi$ for $\boldsymbol{\psi}=\mathbf{0}$ and a vector description $\boldsymbol{v}=\nabla\times\boldsymbol{\psi}$ for $\phi=0$. The cumulative kinematic potentials are decomposed into superpositions of local kinematic potentials

$$\chi=\sum_{n=1}^{N}\chi_n(x,y,z,t),\quad \eta=\sum_{n=1}^{N}\eta_n(x,y,z,t),\quad \psi=\sum_{n=1}^{N}\psi_n(x,y,z,t),\quad \phi=\sum_{n=1}^{N}\phi_n(x,y,z,t). \tag{56}$$

The local kinematic potentials are governed by local Helmholtz PDEs

$$\frac{\partial\psi_n}{\partial y}-\frac{\partial\eta_n}{\partial z}-u_n=0, \tag{57}$$

$$\frac{\partial\chi_n}{\partial z}-\frac{\partial\psi_n}{\partial x}-v_n=0, \tag{58}$$

$$\frac{\partial\eta_n}{\partial x}-\frac{\partial\chi_n}{\partial y}-w_n=0, \tag{59}$$

$$\frac{\partial\chi_n}{\partial x}+\frac{\partial\eta_n}{\partial y}+\frac{\partial\psi_n}{\partial z}=0, \tag{60}$$

$$\frac{\partial\phi_n}{\partial x}-u_n=0, \tag{61}$$

$$\frac{\partial\phi_n}{\partial y}-v_n=0, \tag{62}$$

$$\frac{\partial\varphi_n}{\partial z}-w_n=0, \tag{63}$$

where $n=1,2,\cdots,N$.

The periodic Dirichlet conditions for ψ of the upper and lower cumulative flows are specified on a lower boundary $z=0$ of an upper domain $x\in(-\infty,\infty)$, $y\in(-\infty,\infty)$, $z\in[0,\infty)$ and on the upper boundary $z=0$ of a lower domain $x\in(-\infty,\infty)$, $y\in(-\infty,\infty)$, $z\in(-\infty,0]$ by

$$\psi\big|_{z=0} = \sum_{n=1}^{N}\left(Fs_n cc_n + Qs_n cs_n + Gs_n sc_n + Rs_n ss_n\right), \tag{64}$$

where Fs_n, Qs_n, Gs_n, Rs_n are given boundary coefficients. The experimental solutions show that similar to (28)-(29), the vanishing Dirichlet conditions for ψ and the periodic and vanishing Dirichlet conditions for χ, η, ϕ are redundant.

Construct general terms of the kinematic potentials of the upper and lower flows in the SKF structures by

$$\begin{aligned}
\chi_n &= fh_n(z)cc_n + qh_n(z)cs_n + gh_n(z)sc_n + rh_n(z)ss_n,\\
\eta_n &= fe_n(z)cc_n + qe_n(z)cs_n + ge_n(z)sc_n + re_n(z)ss_n,\\
\psi_n &= fs_n(z)cc_n + qs_n(z)cs_n + gs_n(z)sc_n + rs_n(z)ss_n,\\
\phi_n &= fp_n(z)cc_n + qp_n(z)cs_n + gp_n(z)sc_n + rp_n(z)ss_n.
\end{aligned} \tag{65}$$

Computation of derivatives of (65) by (32)-(34) and substitution in (57)-(63) reduce the seven Helmholtz PDEs to four Helmholtz ODEs and three Helmholtz AEs. For these equations to be satisfied exactly for all independent variables, independent parameters, structural functions, boundary coefficients, and structural coefficients of the upper and lower flows x, y, z, t, Xa_n, Yb_n, Cx_n, Cy_n, ρ_n, σ_n, cc_n, cs_n, sc_n, ss_n, Fw_n, Qw_n, Gw_n, Rw_n, fh_n, qh_n, gh_n, rh_n, fe_n, qe_n, ge_n, re_n, fs_n, qs_n, gs_n, rs_n, fp_n, qp_n, gp_n, rp_n, all coefficients of the kinematic structural functions must vanish. Vanishing four coefficients of seven equations yields 28 equations in total for the upper flows and 28 equations for the lower flows.

For $\boldsymbol{\psi}_n = (\chi_n, \eta_n, \psi_n)$, 16 equations are separated into four systems of four equations each with respect to four groups of the SKF structural coefficients $\left[fh_n(z), re_n(z), gs_n(z)\right]$, $\left[qh_n(z), ge_n(z), rs_n(z)\right]$, $\left[gh_n(z), qe_n(z), fs_n(z)\right]$, $\left[rh_n(z), fe_n(z), qs_n(z)\right]$, for the upper and lower flows, respectively,

$$\begin{aligned}
&-\frac{\mathrm{d}re_n}{\mathrm{d}z} - \sigma_n gs_n \mp \frac{\rho_n}{r_n} Qw_n \exp(\mp r_n z) = 0, \quad \frac{\mathrm{d}fh_n}{\mathrm{d}z} - \rho_n gs_n \pm \frac{\sigma_n}{r_n} Qw_n \exp(\mp r_n z) = 0,\\
&\sigma_n fh_n + \rho_n re_n - Qw_n \exp(\mp r_n z) = 0, \quad -\rho_n fh_n + \sigma_n re_n + \frac{\mathrm{d}gs_n}{\mathrm{d}z} = 0,
\end{aligned} \tag{66}$$

$$\begin{aligned}
&-\frac{\mathrm{d}ge_n}{\mathrm{d}z} + \sigma_n rs_n \mp \frac{\rho_n}{r_n} Fw_n \exp(\mp r_n z) = 0, \quad \frac{\mathrm{d}qh_n}{\mathrm{d}z} - \rho_n rs_n \mp \frac{\sigma_n}{r_n} Fw_n \exp(\mp r_n z) = 0,\\
&-\sigma_n qh_n + \rho_n ge_n - Fw_n \exp(\mp r_n z) = 0, \quad -\rho_n qh_n - \sigma_n ge_n + \frac{\mathrm{d}rs_n}{\mathrm{d}z} = 0,
\end{aligned} \tag{67}$$

$$\begin{aligned}
&-\frac{\mathrm{d}qe_n}{\mathrm{d}z} - \sigma_n fs_n \pm \frac{\rho_n}{r_n} Rw_n \exp(\mp r_n z) = 0, \quad \frac{\mathrm{d}gh_n}{\mathrm{d}z} + \rho_n fs_n \pm \frac{\sigma_n}{r_n} Rw_n \exp(\mp r_n z) = 0,\\
&\sigma_n gh_n - \rho_n qe_n - Rw_n \exp(\mp r_n z) = 0, \quad \rho_n gh_n + \sigma_n qe_n + \frac{\mathrm{d}fs_n}{\mathrm{d}z} = 0,
\end{aligned} \tag{68}$$

$$\begin{aligned}
&-\frac{\mathrm{d}fe_n}{\mathrm{d}z} + \sigma_n qs_n \pm \frac{\rho_n}{r_n} Gw_n \exp(\mp r_n z) = 0, \quad \frac{\mathrm{d}rh_n}{\mathrm{d}z} + \rho_n qs_n \mp \frac{\sigma_n}{r_n} Gw_n \exp(\mp r_n z) = 0,\\
&-\sigma_n rh_n - \rho_n fe_n - Gw_n \exp(\mp r_n z) = 0, \quad \rho_n rh_n - \sigma_n fe_n + \frac{\mathrm{d}qs_n}{\mathrm{d}z} = 0,
\end{aligned} \tag{69}$$

where first and second ODEs are produced by (57)-(58), third AEs are generated by (59), and fourth ODEs are created by (60).

For ϕ_n, 12 equations are separated into four systems of three equations each with respect to four SKF structural coefficients $\left[fp_n(z), qp_n(z), gp_n(z), rp_n(z)\right]$ for the upper and lower flows, respectively,

$$-\rho_n fp_n \mp \frac{\rho_n}{r_n} Fw_n \exp(\mp r_n z) = 0, \quad -\sigma_n fp_n \mp \frac{\sigma_n}{r_n} Fw_n \exp(\mp r_n z) = 0, \quad \frac{\mathrm{d}fp_n}{\mathrm{d}z} - Fw_n \exp(\mp r_n z) = 0, \tag{70}$$

$$-\rho_n qp_n \mp \frac{\rho_n}{r_n} Qw_n \exp(\mp r_n z) = 0, \quad \sigma_n qp_n \pm \frac{\sigma_n}{r_n} Qw_n \exp(\mp r_n z) = 0, \quad \frac{\mathrm{d}qp_n}{\mathrm{d}z} - Qw_n \exp(\mp r_n z) = 0, \tag{71}$$

$$\rho_n gp_n \pm \frac{\rho_n}{r_n} Gw_n \exp(\mp r_n z) = 0, \quad -\sigma_n gp_n \mp \frac{\sigma_n}{r_n} Gw_n \exp(\mp r_n z) = 0, \quad \frac{\mathrm{d}gp_n}{\mathrm{d}z} - Gw_n \exp(\mp r_n z) = 0, \tag{72}$$

$$\rho_n rp_n \pm \frac{\rho_n}{r_n} Rw_n \exp(\mp r_n z) = 0, \quad \sigma_n rp_n \pm \frac{\sigma_n}{r_n} Rw_n \exp(\mp r_n z) = 0, \quad \frac{\mathrm{d}rp_n}{\mathrm{d}z} - Rw_n \exp(\mp r_n z) = 0, \tag{73}$$

where first and second AEs are produced by (61)-(62) and third ODEs are generated by (63).

Solving the third AEs of separated systems (66)-(69) yields functional relations between structural coefficients

$$\begin{aligned} fh_n &= -\frac{\rho_n}{\sigma_n} re_n + \frac{Qw_n}{\sigma_n} \exp(\mp r_n z), \quad qh_n = \frac{\rho_n}{\sigma_n} ge_n - \frac{Fw_n}{\sigma_n} \exp(\mp r_n z), \\ gh_n &= \frac{\rho_n}{\sigma_n} qe_n + \frac{Rw_n}{\sigma_n} \exp(\mp r_n z), \quad rh_n = -\frac{\rho_n}{\sigma_n} fe_n - \frac{Gw_n}{\sigma_n} \exp(\mp r_n z). \end{aligned} \tag{74}$$

Substitution of (74) in the second ODEs of (66)-(69) and addition/subtraction of the first ODEs reduces the second ODEs to identities. Substitution of (74) into the fourth ODEs reduces them to the following system:

$$\begin{aligned} & r_n^2 re_v + \sigma_n \frac{\mathrm{d}gs_n}{\mathrm{d}z} - \rho_n Qw_n \exp(\mp r_n z) = 0, \quad -r_n^2 ge_v + \sigma_n \frac{\mathrm{d}rs_n}{\mathrm{d}z} + \rho_n Fw_n \exp(\mp r_n z) = 0, \\ & r_n^2 qe_v + \sigma_n \frac{\mathrm{d}fs_n}{\mathrm{d}z} + \rho_n Rw_n \exp(\mp r_n z) = 0, \quad -r_n^2 fe_v + \sigma_n \frac{\mathrm{d}qs_n}{\mathrm{d}z} - \rho_n Gw_n \exp(\mp r_n z) = 0. \end{aligned} \tag{75}$$

Solving the first AEs of separated systems (70)-(73) gives structural coefficients

$$fp_n = \mp \frac{Fw_n}{r_n} \exp(\mp r_n z), \quad qp_n = \mp \frac{Qw_n}{r_n} \exp(\mp r_n z), \quad gp_n = \mp \frac{Gw_n}{r_n} \exp(\mp r_n z), \quad rp_n = \mp \frac{Rw_n}{r_n} \exp(\mp r_n z). \tag{76}$$

Substitution of (76) in the second AEs and third ODEs of (70)-(73) reduces them to identities.

Construct solutions of the first ODEs of (66)-(69) and (75) in the SE structures with the following general terms for the upper and lower flows, respectively,

$$(fe_n, qe_n, ge_n, re_n, fs_n, qs_n, gs_n, rs_n)(z) = (Fe_n, Qe_n, Ge_n, Re_n, Fs_n, Qs_n, Gs_n, Rs_n) \exp(\mp r_n z), \tag{77}$$

where Fe_n, Qe_n, Ge_n, Re_n, Fs_n, Qs_n, Gs_n, Rs_n are structural coefficients. Substitution of (77) in the first ODEs of (66)-(69) yields algebraic relations between parameters of the structural coefficients

$$\begin{aligned} Fe_n &= \mp \frac{\sigma_n}{r_n} Qs_n - \frac{\rho_n}{r_n^2} Gw_n, \quad Qe_n = \pm \frac{\sigma_n}{r_n} Fs_n - \frac{\rho_n}{r_n^2} Rw_n, \\ Ge_n &= \mp \frac{\sigma_n}{r_n} Rs_n + \frac{\rho_n}{r_n^2} Fw_n, \quad Re_n = \pm \frac{\sigma_n}{r_n} Gs_n + \frac{\rho_n}{r_n^2} Qw_n. \end{aligned} \tag{78}$$

Substitution of (77) and (78) into (75) reduces them to identities.

Finally, substitutions of (74), (76)-(78) into (65) and (56) give the following kinematic potentials in the SKEF structures for the upper and lower cumulative flows, respectively,

$$\begin{aligned} \chi &= \sum_{n=1}^{N} (Fh_n cc_n + Qh_n cs_n + Gh_n sc_n + Rh_n ss_n) \exp(\mp r_n z), \\ \eta &= \sum_{n=1}^{N} (Fe_n cc_n + Qe_n cs_n + Ge_n sc_n + Rh_n ss_n) \exp(\mp r_n z), \\ \psi &= \sum_{n=1}^{N} (Fs_n cc_n + Qs_n cs_n + Gs_n sc_n + Rs_n ss_n) \exp(\mp r_n z), \\ \phi &= \sum_{n=1}^{N} (Fp_n cc_n + Qp_n cs_n + Gp_n sc_n + Rp_n ss_n) \exp(\mp r_n z), \end{aligned} \tag{79}$$

where structural coefficients are given by (78) and

$$Fh_n = \mp\frac{\rho_n}{r_n}Gs_n + \frac{\sigma_n}{r_n^2}Qw_n, \quad Qh_n = \mp\frac{\rho_n}{r_n}Rs_n - \frac{\sigma_n}{r_n^2}Fw_n,$$
$$Gh_n = \pm\frac{\rho_n}{r_n}Fs_n + \frac{\sigma_n}{r_n^2}Rw_n, \quad Rh_n = \pm\frac{\rho_n}{r_n}Qs_n - \frac{\sigma_n}{r_n^2}Gw_n, \tag{80}$$

$$Fp_n = \mp\frac{1}{r_n}Fw_n, \quad Qp_n = \mp\frac{1}{r_n}Qw_n, \quad Gp_n = \mp\frac{1}{r_n}Gw_n, \quad Rp_n = \mp\frac{1}{r_n}Rw_n. \tag{81}$$

When $\sigma_n = 0$ and $Fs_n = Qs_n = Gs_n = Rs_n = 0$, (78)-(81) are reduced to the 2d solution in the *x*-*z* plane [5]

$$\chi = 0, \quad \eta = \sum_{n=1}^{N}\frac{1}{\rho_n}\left(-Gw_n ca_n + Fw_n sa_n\right)\exp\left(\mp\rho_n z\right),$$
$$\psi = 0, \quad \phi = \mp\sum_{n=1}^{N}\frac{1}{\rho_n}\left(Fw_n ca_n + Gw_n sa_n\right)\exp\left(\mp\rho_n z\right). \tag{82}$$

When $\rho_n = 0$ and $Fs_n = Qs_n = Gs_n = Rs_n = 0$, (78)-(81) are transformed into a 2d solution in the *y*-*z* plane

$$\chi = \sum_{n=1}^{N}\frac{1}{\sigma_n}\left(Qw_n cb_n - Fw_n sb_n\right)\exp\left(\mp\sigma_n z\right), \quad \eta = 0,$$
$$\psi = 0, \quad \phi = \mp\sum_{n=1}^{N}\frac{1}{\sigma_n}\left(Fw_n cb_n + Qw_n sa_n\right)\exp\left(\mp\sigma_n z\right). \tag{83}$$

4. Dynamic Potentials through the SKEF Structures

Definitions of the dynamic vector potential (the Helmholtz potential) $\boldsymbol{h}_\psi = \left(fe, ge, he\right)$ in the vector description $\left(\phi_\psi = 0\right)$ are set by three components of temporal derivative (12)

$$fe = \frac{\partial\chi}{\partial t}, \quad ge = \frac{\partial\eta}{\partial t}, \quad he = \frac{\partial\psi}{\partial t}. \tag{84}$$

Theoretical problems for the dynamic scalar potential (the Bernoulli potential) $b_\psi = be$ in the vector description are set by three components of the global Lamb-Helmholtz PDEs (10)

$$\frac{\partial be}{\partial x} + \frac{\partial he}{\partial y} - \frac{\partial ge}{\partial z} = 0, \quad \frac{\partial be}{\partial y} + \frac{\partial fe}{\partial z} - \frac{\partial he}{\partial x} = 0, \quad \frac{\partial be}{\partial z} + \frac{\partial ge}{\partial x} - \frac{\partial fe}{\partial y} = 0. \tag{85}$$

Since the cumulative dynamic potentials are decomposed into superpositions of local dynamic potentials

$$fe = \sum_{n=1}^{N} fe_n\left(x,y,z,t\right), \quad ge = \sum_{n=1}^{N} ge_n\left(x,y,z,t\right),$$
$$he = \sum_{n=1}^{N} he_n\left(x,y,z,t\right), \quad be = \sum_{n=1}^{N} be_n\left(x,y,z,t\right), \tag{86}$$

the local dynamic potentials are governed by three definitions and three Lamb-Helmholtz PDEs

$$fe_n = \frac{\partial\chi_n}{\partial t}, \quad ge_n = \frac{\partial\eta_n}{\partial t}, \quad he_n = \frac{\partial\psi_n}{\partial t}, \tag{87}$$

$$\frac{\partial be_n}{\partial x} + \frac{\partial he_n}{\partial y} - \frac{\partial ge_n}{\partial z} = 0, \tag{88}$$

$$\frac{\partial be_n}{\partial y} + \frac{\partial fe_n}{\partial z} - \frac{\partial he_n}{\partial x} = 0, \tag{89}$$

$$\frac{\partial be_n}{\partial z} + \frac{\partial ge_n}{\partial x} - \frac{\partial fe_n}{\partial y} = 0, \tag{90}$$

where $n = 1, 2, \cdots, N$. The experimental solutions show that the Dirichlet conditions for fe, ge, he, be are redundant since boundary parameters of $fe,\ ge,\ he,\ be$ depend on the boundary parameters of w and ψ.

Construct a general term of the Bernoulli potential in the SKF structure by

$$be_n = fB_n(z)cc_n + qB_n(z)cs_n + gB_n(z)sc_n + rB_n(z)ss_n \tag{91}$$

and the SKEF structure $l = l(x,y,z,t)$ with a general term l_n and structural coefficients $Fl_n,\ Ql_n,\ Gl_n,\ Rl_n$ as

$$l_n = \left(Fl_n cc_n + Ql_n cs_n + Gl_n sc_n + Rl_n ss_n\right)\exp(\mp r_n z). \tag{92}$$

Computation of temporal and spatial derivatives of l_n yields

$$\begin{aligned}\frac{\partial l_n}{\partial t} = &\left[\left(-Cx_n\rho_n Gl_n - Cy_n\sigma_n Ql_n\right)cc_n + \left(-Cx_n\rho_n Rl_n + Cy_n\sigma_n Fl_n\right)cs_n\right.\\ &\left.+\left(Cx_n\rho_n Fl_n - Cy_n\sigma_n Rl_n\right)sc_n + \left(Cx_n\rho_n Ql_n + Cy_n\sigma_n Gl_n\right)ss_n\right]\exp(\mp r_n z),\end{aligned} \tag{93}$$

$$\frac{\partial l_n}{\partial x} = \rho_n\left(Gl_n cc_n + Rl_n cs_n - Fl_n sc_n - Ql_n ss_n\right)\exp(\mp r_n z), \tag{94}$$

$$\frac{\partial l_n}{\partial y} = \sigma_n\left(Ql_n cc_n - Fl_n cs_n + Rl_n sc_n - Gl_n ss_n\right)\exp(\mp r_n z), \tag{95}$$

$$\frac{\partial l_n}{\partial z} = \mp r_n\left(Fl_n cc_n + Ql_n cs_n + Gl_n sc_n + Rl_n ss_n\right)\exp(\mp r_n z). \tag{96}$$

Application of (93) to (87) and (79) gives the following Helmholtz potentials in the SKEF structures for the upper and lower cumulative flows, respectively,

$$\begin{aligned}fe &= \sum_{n=1}^{N}\left(FF_n cc_n + QF_n cs_n + GF_n sc_n + RF_n ss_n\right)\exp(\mp r_n z),\\ ge &= \sum_{n=1}^{N}\left(FG_n cc_n + QG_n cs_n + GG_n sc_n + RG_n ss_n\right)\exp(\mp r_n z),\\ he &= \sum_{n=1}^{N}\left(FH_n cc_n + QH_n cs_n + GH_n sc_n + RH_n ss_n\right)\exp(\mp r_n z),\end{aligned} \tag{97}$$

where structural coefficients are

$$\begin{aligned}FF_n &= -Cx_n\rho_n Gh_n - Cy_n\sigma_n Qh_n, \quad QF_n = -Cx_n\rho_n Rh_n + Cy_n\sigma_n Fh_n,\\ GF_n &= Cx_n\rho_n Fh_n - Cy_n\sigma_n Rh_n, \quad RF_n = Cx_n\rho_n Qh_n + Cy_n\sigma_n Gh_n,\end{aligned} \tag{98}$$

$$\begin{aligned}FG_n &= -Cx_n\rho_n Ge_n - Cy_n\sigma_n Qe_n, \quad QG_n = -Cx_n\rho_n Re_n + Cy_n\sigma_n Fe_n,\\ GG_n &= Cx_n\rho_n Fe_n - Cy_n\sigma_n Re_n, \quad RG_n = Cx_n\rho_n Qe_n + Cy_n\sigma_n Ge_n,\end{aligned} \tag{99}$$

$$\begin{aligned}FH_n &= -Cx_n\rho_n Gs_n - Cy_n\sigma_n Qs_n, \quad QH_n = -Cx_n\rho_n Rs_n + Cy_n\sigma_n Fs_n,\\ GH_n &= Cx_n\rho_n Fs_n - Cy_n\sigma_n Rs_n, \quad RH_n = Cx_n\rho_n Qs_n + Cy_n\sigma_n Gs_n.\end{aligned} \tag{100}$$

Computation of derivatives of (97) and (91) by (94)-(96) and (32)-(34), respectively, and substitution in (88)-(90) reduce the three Lamb-Helmholtz PDEs to two Lamb-Helmholtz AEs and one Lamb-Helmholtz ODE. For these equations to be satisfied exactly for all independent variables, independent parameters, structural functions, boundary coefficients, and structural coefficients of the upper and lower flows $x,\ y,\ z,\ t,\ Xa_n,\ Yb_n,\ Cx_n,\ Cy_n,\ \rho_n,\ \sigma_n,\ cc_n,\ cs_n,\ sc_n,\ ss_n,\ FF_n,\ QF_n,\ GF_n,\ RF_n,\ FG_n,\ QG_n,\ GG_n,\ RG_n,\ FH_n,\ QH_n,\ GH_n,\ RH_n,\ fB_n,\ qB_n,\ gB_n,\ rB_n$, all coefficients of the kinematic structural functions must vanish. Vanishing four coefficients of three equations yields 12 equations in total for the upper flows and 12 equations for the lower flows, which are separated into four systems of three equations each with respect to four SKF structural coefficients $\left[fB_n(z), qB_n(z), gG_n(z), rB_n(z)\right]$ for the upper and lower flows, respectively,

$$\begin{aligned}&-\rho_n fB_n + \left(\pm r_n GG_n + \sigma_n RH_n\right)\exp(\mp r_n z) = 0, \quad -\sigma_n fB_n + \left(\mp r_n QF_n - \rho_n RH_n\right)\exp(\mp r_n z) = 0,\\ &\frac{\mathrm{d}fB_n}{\mathrm{d}z} + \left(\rho_n GG_n - \sigma_n QF_n\right)\exp(\mp r_n z) = 0,\end{aligned} \tag{101}$$

$$-\rho_n qB_n + \left(\pm r_n RG_n - \sigma_n GH_n\right)\exp\left(\mp r_n z\right) = 0, \quad \sigma_n qB_n + \left(\mp r_n FF_n - \rho_n GH_n\right)\exp\left(\mp r_n z\right) = 0,$$
$$\frac{\mathrm{d}qB_n}{\mathrm{d}z} + \left(\rho_n RG_n + \sigma_n FF_n\right)\exp\left(\mp r_n z\right) = 0, \tag{102}$$

$$\rho_n gB_n + \left(\pm r_n FG_n + \sigma_n QH_n\right)\exp\left(\mp r_n z\right) = 0, \quad -\sigma_n gB_n + \left(\mp r_n RF_n + \rho_n QH_n\right)\exp\left(\mp r_n z\right) = 0,$$
$$\frac{\mathrm{d}gB_n}{\mathrm{d}z} - \left(\rho_n FG_n + \sigma_n RF_n\right)\exp\left(\mp r_n z\right) = 0, \tag{103}$$

$$\rho_n rB_n + \left(\pm r_n QG_n - \sigma_n FH_n\right)\exp\left(\mp r_n z\right) = 0, \quad \sigma_n rB_n + \left(\mp r_n GF_n + \rho_n FH_n\right)\exp\left(\mp r_n z\right) = 0,$$
$$\frac{\mathrm{d}rB_n}{\mathrm{d}z} + \left(-\rho_n QG_n + \sigma_n GF_n\right)\exp\left(\mp r_n z\right) = 0, \tag{104}$$

where first and second AEs are produced by (88)-(89) and third ODEs are generated by (90).

Solving the first AEs of separated systems (101)-(104) gives the following Bernoulli potentials in the SKEF structures for the upper and lower cumulative flows, respectively,

$$be = \sum_{n=1}^{N}\left(FB_n cc_n + QB_n cs_n + GB_n sc_n + RB_n ss_n\right)\exp\left(\mp r_n z\right), \tag{105}$$

where structural coefficients are

$$FB_n = \pm\left(-Cx_n \frac{\rho_n}{r_n} Gw_n - Cy_n \frac{\sigma_n}{r_n} Qw_n\right), \quad QB_n = \pm\left(-Cx_n \frac{\rho_n}{r_n} Rw_n + Cy_n \frac{\sigma_n}{r_n} Fw_n\right),$$
$$GB_n = \pm\left(Cx_n \frac{\rho_n}{r_n} Fw_n - Cy_n \frac{\sigma_n}{r_n} Rw_n\right), \quad RB_n = \pm\left(Cx_n \frac{\rho_n}{r_n} Qw_n + Cy_n \frac{\sigma_n}{r_n} Gw_n\right). \tag{106}$$

Substitution of (106) in the second AEs and third ODEs of (101)-(104) reduces them to identities. The Bernoulli potential (105)-(106) does not depend on boundary coefficients of the kinematic potentials Fs_n, Qs_n, Gs_n, Rs_n since ψ of (79) generates the SKEF solution of the homogeneous problem $\nabla\times\boldsymbol{\psi} = \mathbf{0}$.

In the scalar description, $\boldsymbol{\psi}_\phi = 0$, $\boldsymbol{h}_\phi = \mathbf{0}$ and integration of the Lamb-Helmholtz PDE (10) returns the Bernoulli equation [2]

$$be_\phi = \frac{\partial\phi}{\partial t} + p_d + k_e + f\left(t\right) = 0, \tag{107}$$

where $\phi = \phi_\phi$ and an arbitrary function of time $f\left(t\right)$ vanishes for the SKEF structures because of the vanishing Dirichlet conditions (24) and (26). Using the scalar-vector duality, the Lamb-Helmholtz PDE (10) reads

$$\nabla b_\psi + \nabla\times\boldsymbol{h}_\psi = \nabla be + \nabla\times\frac{\partial\boldsymbol{\psi}}{\partial t} = \nabla be + \frac{\partial}{\partial t}\nabla\times\boldsymbol{\psi} = \nabla be + \frac{\partial\mathbf{v}}{\partial t} = \nabla be + \frac{\partial}{\partial t}\nabla\phi = \nabla\left(be + \frac{\partial\phi}{\partial t}\right) = \mathbf{0}. \tag{108}$$

where $\boldsymbol{\psi} = \boldsymbol{\psi}_\psi$. Integration of (108) returns dual formulas for the global and local Bernoulli potentials

$$be = -\frac{\partial\phi}{\partial t}, \tag{109}$$

$$be_n = -\frac{\partial\phi_n}{\partial t} \tag{110}$$

since an integration constant again vanishes for the SKEF structures in agreement with (24) and (26). Therefore, computation of *be* by (110), (79), and (93) also results in (105)-(106).

5. Stationary Dynamic Exponential Fourier (SDEF) Structures

By the generalized Einstein notation for summation that is extended for exponents in [5], define two SKEF structures $l\left(x,y,z,t\right) = l$ and $h\left(x,y,z,t\right) = h$ with general terms l_n and h_m for the upper and lower cumulative flows, respectively,

$$l = \sum_{n=1}^{N} l_n(x,y,z,t) = \left(Fl_n cc_n + Ql_n cs_n + Gl_n sc_n + Rl_n ss_n\right)\exp\left(\mp r_n z\right),$$
$$h = \sum_{m=1}^{N} h_m(x,y,z,t) = \left(Fh_m cc_m + Qh_m cs_m + Gh_m sc_m + Rh_m ss_m\right)\exp\left(\mp r_n z\right). \tag{111}$$

Following the experimental solutions, construct 16 trigonometric structural functions of the SDEF structure

$$\begin{aligned}
CdCd_{n,m} &= \cos(\alpha_n - \alpha_m)\cos(\beta_n - \beta_m), \quad CdCs_{n,m} = \cos(\alpha_n - \alpha_m)\cos(\beta_n + \beta_m),\\
CdSd_{n,m} &= \cos(\alpha_n - \alpha_m)\sin(\beta_n - \beta_m), \quad CdSs_{n,m} = \cos(\alpha_n - \alpha_m)\sin(\beta_n + \beta_m),\\
CsCd_{n,m} &= \cos(\alpha_n + \alpha_m)\cos(\beta_n - \beta_m), \quad CsCs_{n,m} = \cos(\alpha_n + \alpha_m)\cos(\beta_n + \beta_m),\\
CsSd_{n,m} &= \cos(\alpha_n + \alpha_m)\sin(\beta_n - \beta_m), \quad CsSs_{n,m} = \cos(\alpha_n + \alpha_m)\sin(\beta_n + \beta_m),\\
SdCd_{n,m} &= \sin(\alpha_n - \alpha_m)\cos(\beta_n - \beta_m), \quad SdCs_{n,m} = \sin(\alpha_n - \alpha_m)\cos(\beta_n + \beta_m),\\
SdSd_{n,m} &= \sin(\alpha_n - \alpha_m)\sin(\beta_n - \beta_m), \quad SdSs_{n,m} = \sin(\alpha_n - \alpha_m)\sin(\beta_n + \beta_m),\\
SsCd_{n,m} &= \sin(\alpha_n + \alpha_m)\cos(\beta_n - \beta_m), \quad SsCs_{n,m} = \sin(\alpha_n + \alpha_m)\cos(\beta_n + \beta_m),\\
SsSd_{n,m} &= \sin(\alpha_n + \alpha_m)\sin(\beta_n - \beta_m), \quad SsSs_{n,m} = \sin(\alpha_n + \alpha_m)\sin(\beta_n + \beta_m),
\end{aligned} \tag{112}$$

where capital letters C and S stand for cosine and sine, letters s and d for sum and difference of arguments α_n, α_m and β_n, β_m.

Computation of a general term $p_{n,n} = l_n h_n$ of product $p(x,y,z,t) = p = lh$ by one-dimensional summation of diagonal terms yields for the upper and lower flows, respectively,

$$\begin{aligned}
p_{n,n} = \frac{1}{4}\big(&Fddp_{n,n} + Fdsp_{n,n}CdCs_{n,n} + Qdsp_{n,n}CdSs_{n,n} + Fsdp_{n,n}CsCd_{n,n} + Fssp_{n,n}CsCs_{n,n}\\
&+ Qssp_{n,n}CsSs_{n,n} + Gsdp_{n,n}SsCd_{n,n} + Gssp_{n,n}SsCs_{n,n} + Rssp_{n,n}SsSs_{n,n}\big)\exp\left(\mp 2r_n z\right),
\end{aligned} \tag{113}$$

where structural coefficients are

$$\begin{aligned}
Fddp_{n,n} &= Fl_nFh_n + Ql_nQh_n + Gl_nGh_n + Rl_nRh_n, \quad Fdsp_{n,n} = Fl_nFh_n - Ql_nQh_n + Gl_nGh_n - Rl_nRh_n,\\
Qdsp_{n,n} &= Fl_nQh_n + Fh_nQl_n + Gl_nRh_n + Gh_nRl_n, \quad Fsdp_{n,n} = Fl_nFh_n + Ql_nQh_n - Gl_nGh_n - Rl_nRh_n,\\
Fssp_{n,n} &= Fl_nFh_n - Ql_nQh_n - Gl_nGh_n + Rl_nRh_n, \quad Qssp_{n,n} = Fl_nQh_n + Fh_nQl_n - Gl_nRh_n - Gh_nRl_n,\\
Gsdp_{n,n} &= Fl_nGh_n + Fh_nGl_n + Ql_nRh_n + Qh_nRl_n, \quad Gssp_{n,n} = Fl_nGh_n + Fh_nGl_n - Ql_nRh_n - Qh_nRl_n,\\
Rssp_{n,n} &= Fl_nRh_n + Fh_nRl_n + Ql_nGh_n + Qh_nGl_n.
\end{aligned} \tag{114}$$

A general term $p_{n,m} = l_n h_m$ of product p computed by rectangular summation of non-diagonal terms for the upper and lower flows, respectively, becomes

$$\begin{aligned}
p_{n,m} = \frac{1}{4}\big(&Fddp_{n,m}CdCd_{n,m} + Fdsp_{n,m}CdCs_{n,m} + Qddp_{n,m}CdSd_{n,m} + Qdsp_{n,m}CdSs_{n,m} + Fsdp_{n,m}CsCd_{n,m}\\
&+ Fssp_{n,m}CsCs_{n,m} + Qsdp_{n,m}CsSd_{n,m} + Qssp_{n,m}CsSs_{n,m} + Gddp_{n,m}SdCd_{n,m} + Gdsp_{n,m}SdCs_{n,m}\\
&+ Rddp_{n,m}SdSd_{n,m} + Rdsp_{n,m}SdSs_{n,m} + Gsdp_{n,m}SsCd_{n,m} + Gssp_{n,m}SsCs_{n,m} + Rsdp_{n,m}SsSd_{n,m}\\
&+ Rssp_{n,m}SsSs_{n,m}\big)\exp\left[\mp(r_n + r_m)z\right],
\end{aligned} \tag{115}$$

where structural coefficients are

$$\begin{aligned}
Fddp_{n,m} &= Fl_nFh_m + Ql_nQh_m + Gl_nGh_m + Rl_nRh_m, \quad Fdsp_{n,m} = Fl_nFh_m - Ql_nQh_m + Gl_nGh_m - Rl_nRh_m,\\
Qddp_{n,m} &= -Fl_nQh_m + Fh_mQl_n - Gl_nRh_m + Gh_mRl_n, \quad Qdsp_{n,m} = Fl_nQh_m + Fh_mQl_n + Gl_nRh_m + Gh_mRl_n,\\
Fsdp_{n,m} &= Fl_nFh_m + Ql_nQh_m - Gl_nGh_m - Rl_nRh_m, \quad Fssp_{n,m} = Fl_nFh_m - Ql_nQh_m - Gl_nGh_m + Rl_nRh_m,\\
Qsdp_{n,m} &= -Fl_nQh_m + Fh_mQl_n + Gl_nRh_m - Gh_mRl_n, \quad Qssp_{n,m} = Fl_nQh_m + Fh_mQl_n - Gl_nRh_m - Gh_mRl_n,\\
Gddp_{n,m} &= -Fl_nGh_m + Fh_mGl_n - Ql_nRh_m + Qh_mRl_n, \quad Gdsp_{n,m} = -Fl_nGh_m + Fh_mGl_n + Ql_nRh_m - Qh_mRl_n,\\
Rddp_{n,m} &= Fl_nRh_m + Fh_mRl_n - Ql_nGh_m - Qh_mGl_n, \quad Rdsp_{n,m} = -Fl_nRh_m + Fh_mRl_n - Ql_nGh_m + Qh_mGl_n,\\
Gsdp_{n,m} &= Fl_nGh_m + Fh_mGl_n + Ql_nRh_m + Qh_mRl_n, \quad Gssp_{n,m} = Fl_nGh_m + Fh_mGl_n - Ql_nRh_m - Qh_mRl_n,\\
Rsdp_{n,m} &= -Fl_nRh_m + Fh_mRl_n + Ql_nGh_m - Qh_mGl_n, \quad Rsdp_{n,m} = Fl_nRh_m + Fh_mRl_n + Ql_nGh_m + Qh_mGl_n.
\end{aligned} \tag{116}$$

Conversion of (115)-(116) by triangular summation of non-diagonal terms and addition of (113)-(114) yields summation formulas for the product of the SKEF structures written through the SDEF structures for the upper and lower flows, respectively,

$$\begin{aligned} p = & \frac{1}{4}\sum_{n=1}^{N}\big(Fddp_{n,n} + Fdsp_{n,n}CdCs_{n,n} + Qdsp_{n,n}CdSs_{n,n} + Fsdp_{n,n}CsCd_{n,n} + Fssp_{n,n}CsCs_{n,n} \\ & + Qssp_{n,n}CsSs_{n,n} + Gsdp_{n,n}ScCd_{n,n} + Gssp_{n,n}ScCs_{n,n} + Rssp_{n,n}SsSs_{n,n}\big)\exp(\mp 2r_n z) \\ & + \frac{1}{4}\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\big(Fddp_{n,m}CdCd_{n,m} + Fdsp_{n,m}CdCs_{n,m} + Qddp_{n,m}CdSd_{n,m} + Qdsp_{n,m}CdSs_{n,m} \\ & + Fsdp_{n,m}CsCd_{n,m} + Fssp_{n,m}CsCs_{n,m} + Qsdp_{n,m}CsSd_{n,m} + Qssp_{n,m}CsSs_{n,m} + Gddp_{n,m}SdCd_{n,m} \\ & + Gdsp_{n,m}SdCs_{n,m} + Rddp_{n,m}SdSd_{n,m} + Rdsp_{n,m}SdSs_{n,m} + Gsdp_{n,m}SsCd_{n,m} + Gssp_{n,m}SsCs_{n,m} \\ & + Rsdp_{n,m}SsSd_{n,m} + Rssp_{n,m}SsSs_{n,m}\big)\exp\big[\mp(r_n + r_m)z\big], \end{aligned} \tag{117}$$

where 9 diagonal structural coefficients are given by (114) and 16 non-diagonal structural coefficients are

$$\begin{aligned} Fddp_{n,m} &= Fl_nFh_m + Fl_mFh_n + Ql_nQh_m + Ql_mQh_n + Gl_nGh_m + Gl_mGh_n + Rl_nRh_m + Rl_mRh_n, \\ Fdsp_{n,m} &= Fl_nFh_m + Fl_mFh_n - Ql_nQh_m - Ql_mQh_n + Gl_nGh_m + Gl_mGh_n - Rl_nRh_m - Rl_mRh_n, \\ Qddp_{n,n} &= -Fl_nQh_m + Fl_mQh_n - Fh_nQl_m + Fh_mQl_n - Gl_nRh_m + Gl_mRh_n - Gh_nRl_m + Gh_mRl_n, \\ Qdsp_{n,m} &= Fl_nQh_m + Fl_mQh_n + Fh_nQl_m + Fh_mQl_n + Gl_nRh_m + Gl_mRh_n + Gh_nRl_m + Gh_mRl_n, \\ Fsdp_{n,m} &= Fl_nFh_m + Fl_mFh_n + Ql_nQh_m + Ql_mQh_n - Gl_nGh_m - Gl_mGh_n - Rl_nRh_m - Rl_mRh_n, \\ Fssp_{n,n} &= Fl_nFh_m + Fl_mFh_n - Ql_nQh_m - Ql_mQh_n - Gl_nGh_m - Gl_mGh_n + Rl_nRh_m + Rl_mRh_n, \\ Qsdp_{n,m} &= -Fl_nQh_m + Fl_mQh_n - Fh_nQl_m + Fh_mQl_n + Gl_nRh_m - Gl_mRh_n + Gh_nRl_m - Gh_mRl_n, \\ Qssp_{n,m} &= Fl_nQh_m + Fl_mQh_n + Fh_nQl_m + Fh_mQl_n - Gl_nRh_m - Gl_mRh_n - Gh_nRl_m - Gh_mRl_n, \\ Gddp_{n,m} &= -Fl_nGh_m + Fl_mGh_n - Fh_nGl_m + Fh_mGl_n - Ql_nRh_m + Ql_mRh_n - Qh_nRl_m + Qh_mRl_n, \\ Gdsp_{n,m} &= -Fl_nGh_m + Fl_mGh_n - Fh_nGl_m + Fh_mGl_n + Ql_nRh_m - Ql_mRh_n + Qh_nRl_m - Qh_mRl_n, \\ Rddp_{n,m} &= Fl_nRh_m + Fl_mRh_n + Fh_nRl_m + Fh_mRl_n - Ql_nGh_m - Ql_mGh_n - Qh_nGl_m - Qh_mGl_n, \\ Rdsp_{n,m} &= -Fl_nRh_m + Fl_mRh_n - Fh_nRl_m + Fh_mRl_n - Ql_nGh_m + Ql_mGh_n - Qh_nGl_m + Qh_mGl_n, \\ Gsdp_{n,m} &= Fl_nGh_m + Fl_mGh_n + Fh_nGl_m + Fh_mGl_n + Ql_nRh_m + Ql_mRh_n + Qh_nRl_m + Qh_mRl_n, \\ Gssp_{n,m} &= Fl_nGh_m + Fl_mGh_n + Fh_nGl_m + Fh_mGl_n - Ql_nRh_m - Ql_mRh_n - Qh_nRl_m - Qh_mRl_n, \\ Rsdp_{n,m} &= -Fl_nRh_m + Fl_mRh_n - Fh_nRl_n + Fh_mRl_n + Ql_nGh_m - Ql_mGh_n + Qh_nGl_m - Qh_mGl_n, \\ Rsdp_{n,m} &= Fl_nRh_m + Fl_mRh_n + Fh_nRl_m + Fh_mRl_n + Ql_nGh_m + Ql_mGh_n + Qh_nGl_m + Qh_mGl_n. \end{aligned} \tag{118}$$

6. Total Pressures through the SKEF-SDEF Structures

The kinetic energy per unit mass of the upper and lower cumulative flows,

$$k_e(x, y, z, t) = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}(u\cdot u + v\cdot v + w\cdot w), \tag{119}$$

is computed as a superposition of three products of the SKEF structures for the velocity components that are converted from the functional form (44) to the structural form (92) for the upper and lower cumulative flows, respectively, as

$$\begin{aligned} u &= \sum_{n=1}^{N}\big(Fu_ncc_n + Qu_ncs_n + Gu_nsc_n + Ru_nss_n\big)\exp(\mp r_n z), \\ v &= \sum_{n=1}^{N}\big(Fv_ncc_n + Qv_ncs_n + Gv_nsc_n + Rv_nss_n\big)\exp(\mp r_n z), \\ w &= \sum_{n=1}^{N}\big(Fw_ncc_n + Qw_ncs_n + Gw_nsc_n + Rw_nss_n\big)\exp(\mp r_n z), \end{aligned} \tag{120}$$

where the structural coefficients for the upper and lower flows, respectively, are

$$\begin{aligned} &Fu_n = \mp C_n Gw_n, \quad Qu_n = \mp C_n Rw_n, \quad Gu_n = \pm C_n Fw_n, \quad Ru_n = \pm C_n Qw_n, \\ &Fv_n = \mp S_n Qw_n, \quad Qv_n = \pm S_n Fw_n, \quad Gv_n = \mp S_n Rw_n, \quad Rv_n = \pm S_n Gw_n, \end{aligned} \tag{121}$$

structural parameters C_n and S_n are given by (47). Application of the product rules (117)-(118) for transformation of the SKEF structures into the SDEF structures to the cumulative kinetic energy of the upper and lower flows, respectively, yields

$$\begin{aligned} k_e &= \frac{1}{4}\sum_{n=1}^{N}\left(Fddk_{n,n} + Fdsk_{n,n}CdCs_{n,n} + Qdsk_{n,n}CdSs_{n,n} + Fsdk_{n,n}CsCd_{n,n} + Gsdk_{n,n}ScCd_{n,n}\right)\exp\left(\mp 2r_n z\right) \\ &+ \frac{1}{4}\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\left(Fddk_{n,m}CdCd_{n,m} + Fdsk_{n,m}CdCs_{n,m} + Qddk_{n,m}CdSd_{n,m} + Qdsk_{n,m}CdSs_{n,m}\right. \\ &+ Fsdk_{n,m}CsCd_{n,m} + Fssk_{n,m}CsCs_{n,m} + Qsdk_{n,m}CsSd_{n,m} + Qssk_{n,m}CsSs_{n,m} + Gddk_{n,m}SdCd_{n,m} \\ &+ Gdsk_{n,m}SdCs_{n,m} + Rddk_{n,m}SdSd_{n,m} + Rdsk_{n,m}SdSs_{n,m} + Gsdk_{n,m}SsCd_{n,m} + Gssk_{n,m}SsCs_{n,m} \\ &\left.+ Rsdk_{n,m}SsSd_{n,m} + Rssk_{n,m}SsSs_{n,m}\right)\exp\left[\mp\left(r_n + r_m\right)z\right], \end{aligned} \tag{122}$$

where the structural coefficients are

$$\begin{aligned} Fddk_{n,n} &= Fw_n^2 + Qw_n^2 + Gw_n^2 + Rw_n^2, \\ Fdsk_{n,n} &= C_n^2\left(Fw_n^2 - Qw_n^2 + Gw_n^2 - Rw_n^2\right), \quad Qdsk_{n,n} = 2C_n^2\left(Fw_nQw_n + Gw_nRw_n\right), \\ Fsdk_{n,n} &= S_n^2\left(Fw_n^2 + Qw_n^2 - Gw_n^2 - Rw_n^2\right), \quad Gdsk_{n,n} = 2S_n^2\left(Fw_nGw_n + Qw_nRw_n\right), \\ Fddk_{n,m} &= \left(1+Td_{n,m}\right)\left(Fw_nFw_m + Qw_nQw_m + Gw_nGw_m + Rw_nRw_m\right), \\ Fdsk_{n,m} &= \left(1+Ts_{n,m}\right)\left(Fw_nFw_m - Qw_nQw_m + Gw_nGw_m - Rw_nRw_m\right), \\ Qddk_{n,m} &= \left(1+Td_{n,m}\right)\left(-Fw_nQw_m + Fw_mQw_n - Gw_nRw_m + Gw_mRw_n\right), \\ Qdsk_{n,m} &= \left(1+Ts_{n,m}\right)\left(Fw_nQw_m + Fw_mQw_n + Gw_nRw_m + Gw_mRw_n\right), \\ Fsdk_{n,m} &= \left(1-Ts_{n,m}\right)\left(Fw_nFw_m + Qw_nQw_m - Gw_nGw_m - Rw_nRw_m\right), \\ Fssk_{n,m} &= \left(1-Td_{n,m}\right)\left(Fw_nFw_m - Qw_nQw_m - Gw_nGw_m + Rw_nRw_m\right), \\ Qsdk_{n,m} &= \left(1-Ts_{n,m}\right)\left(-Fw_nQw_m + Fw_mQw_n + Gw_nRw_m - Gw_mRw_n\right), \\ Qssk_{n,m} &= \left(1-Td_{n,m}\right)\left(Fw_nQw_m + Fw_mQw_n - Gw_nRw_m - Gw_mRw_n\right), \\ Gddk_{n,m} &= \left(1+Td_{n,m}\right)\left(-Fw_nGw_m + Fw_mGw_n - Qw_nRw_m + Qw_mRw_n\right), \\ Gdsk_{n,m} &= \left(1+Ts_{n,m}\right)\left(-Fw_nGw_m + Fw_mGw_n + Qw_nRw_m - Qw_mRw_n\right), \\ Rddk_{n,m} &= \left(1+Td_{n,m}\right)\left(Fw_nRw_m + Fw_mRw_n - Qw_nGw_m - Qw_mGw_n\right), \\ Rdsk_{n,m} &= \left(1+Ts_{n,m}\right)\left(-Fw_nRw_m + Fw_mRw_n - Qw_nGw_m + Qw_mGw_n\right), \\ Gsdk_{n,m} &= \left(1-Ts_{n,m}\right)\left(Fw_nGw_m + Fw_mGw_n + Qw_nRw_m + Qw_mRw_n\right), \\ Gssk_{n,m} &= \left(1-Td_{n,m}\right)\left(Fw_nGw_m + Fw_mGw_n - Qw_nRw_m - Qw_mRw_n\right), \\ Rsdk_{n,m} &= \left(1-Ts_{n,m}\right)\left(-Fw_nRw_m + Fw_mRw_n + Qw_nGw_m - Qw_mGw_n\right), \\ Rssk_{n,m} &= \left(1-Td_{n,m}\right)\left(Fw_nRw_m + Fw_mRw_n + Qw_nGw_m + Qw_mGw_n\right). \end{aligned} \tag{123}$$

Here, structural parameters of energy pulsations $Td_{n,m}$ and $Ts_{n,m}$ in the algebraic and trigonometric forms are

$$Td_{n,m} = C_nC_m + S_nS_m = \cos\left(\theta_n - \theta_m\right), \quad Ts_{n,m} = C_nC_m - S_nS_m = \cos\left(\theta_n + \theta_m\right). \tag{124}$$

Since the velocity components have a unique presentation both for the vector and scalar descriptions of the kinematic and dynamic potentials, the kinetic energies also have a single description.

Substitution of the dual formula (109) for the Bernoulli potential *be* into the Bernoulli Equation (11) in the vector description returns the same expression for the dynamic pressure as the Bernoulli Equation in the scalar description (107)

$$p_d(x,y,z,t) = be(x,y,z,t) - k_e(x,y,z,t). \tag{125}$$

Thus, the kinetic energies, the dynamic pressures, and the total pressures have a unique presentation both in the vector and scalar descriptions.

Substitution of the dynamic pressure in the hydrostatic Equation (5) yields the total pressure for the upper and lower cumulative flows, respectively,

$$\begin{aligned} p_t = {} & p_0 - \rho g_z z + \rho \sum_{n=1}^{N} \left(FB_n cc_n + QB_n cs_n + GB_n sc_n + RB_n ss_n \right) \exp(\mp r_n z) \\ & - \frac{\rho}{4} \sum_{n=1}^{N} \left(Fddk_{n,n} + Fdsk_{n,n} CdCs_{n,n} + Qdsk_{n,n} CdSs_{n,n} + Fsdk_{n,n} CsCd_{n,n} + Gsdk_{n,n} ScCd_{n,n} \right) \exp(\mp 2 r_n z) \\ & - \frac{\rho}{4} \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \left(Fddk_{n,m} CdCd_{n,m} + Fdsk_{n,m} CdCs_{n,m} + Qddk_{n,m} CdSd_{n,m} + Qdsk_{n,m} CdSs_{n,m} \right. \\ & + Fsdk_{n,m} CsCd_{n,m} + Fssk_{n,m} CsCs_{n,m} + Qsdk_{n,m} CsSd_{n,m} + Qssk_{n,m} CsSs_{n,m} + Gddk_{n,m} SdCd_{n,m} \\ & + Gdsk_{n,m} SdCs_{n,m} + Rddk_{n,m} SdSd_{n,m} + Rdsk_{n,m} SdSs_{n,m} + Gsdk_{n,m} SsCd_{n,m} + Gssk_{n,m} SsCs_{n,m} \\ & \left. + Rsdk_{n,m} SsSd_{n,m} + Rssk_{n,m} SsSs_{n,m} \right) \exp\left[\mp (r_n + r_m) z \right], \end{aligned} \tag{126}$$

where structural coefficients are given by (106) and (123).

When $\sigma_n = 0$, (126) is reduced to the 2d solution in the x-z plane [5]

$$\begin{aligned} p_t = {} & p_0 - \rho g_z z \mp \rho \sum_{n=1}^{N} Cx_n \left(Gw_n ca_n - Fw_n sa_n \right) \exp(\mp \rho_n z) - \frac{\rho}{2} \sum_{n=1}^{N} \left(Fw_n^2 + Gw_n^2 \right) \exp(\mp 2 \rho_n z) \\ & - \rho \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \left[\left(Fw_n Fw_m + Gw_n Gw_m \right) Cad_{n,m} + \left(-Fw_n Gw_m + Fw_m Gw_n \right) Sad_{n,m} \right] \exp\left[\mp (\rho_n + \rho_m) z \right]. \end{aligned} \tag{127}$$

When $\rho_n = 0$, (126) is converted into the 2d solution in the y-z plane

$$\begin{aligned} p_t = {} & p_0 - \rho g_z z \mp \rho \sum_{n=1}^{N} Cy_n \left(Qw_n cb_n - Fw_n sb_n \right) \exp(\mp \sigma_n z) - \frac{\rho}{2} \sum_{n=1}^{N} \left(Fw_n^2 + Qw_n^2 \right) \exp(\mp 2 \sigma_n z) \\ & - \rho \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \left[\left(Fw_n Fw_m + Qw_n Qw_m \right) Cbd_{n,m} + \left(-Fw_n Qw_m + Fw_m Qw_n \right) Sbd_{n,m} \right] \exp\left[\mp (\sigma_n + \sigma_m) z \right]. \end{aligned} \tag{128}$$

In (127)-(128), $Cad_{n,m} = \cos(\alpha_n - \alpha_m)$, $Sad_{n,m} = \sin(\alpha_n - \alpha_m)$, $Cbd_{n,m} = \cos(\beta_n - \beta_m)$, $Sbd_{n,m} = \sin(\beta_n - \beta_m)$.

7. Decomposition of Harmonic Variables in the SKEF Structural Basis

Similar to two independent SKEF structures in two dimensions [5], there are four independent SKEF structures in three dimensions

$$\begin{aligned} Ah &= \sum_{n=1}^{N} Ah_n(x,y,z,t) = \sum_{n=1}^{N} \left(Fa_n cc_n + Qa_n cs_n + Ga_n sc_n + Ra_n ss_n \right) \exp(\mp r_n z), \\ Bh &= \sum_{n=1}^{N} Bh_n(x,y,z,t) = \sum_{n=1}^{N} \left(Ga_n cc_n + Ra_n cs_n - Fa_n sc_n - Qa_n ss_n \right) \exp(\mp r_n z), \\ Ch &= \sum_{n=1}^{N} Ch_n(x,y,z,t) = \sum_{n=1}^{N} \left(Qa_n cc_n - Fa_n cs_n + Ra_n sc_n - Ga_n ss_n \right) \exp(\mp r_n z), \\ Dh &= \sum_{n=1}^{N} Dh_n(x,y,z,t) = \sum_{n=1}^{N} \left(Ra_n cc_n - Ga_n cs_n - Qa_n sc_n + Fa_n ss_n \right) \exp(\mp r_n z), \end{aligned} \tag{129}$$

where Fa_n, Qa_n, Ga_n, Ra_n are structural coefficients. Computation of the spatial and temporal derivatives of the general terms of (129) by (93)-(96) yields for the upper and lower flows, respectively,

$$\begin{aligned}
&\frac{\partial Ah_n}{\partial x}=\rho_n Bh_n, \quad \frac{\partial Ah_n}{\partial y}=\sigma_n Ch_n, \quad \frac{\partial Ah_n}{\partial z}=\mp r_n Ah_n, \quad \frac{\partial Ah_n}{\partial t}=-Cx_n\rho_n Bh_n-Cy_n\sigma_n Ch_n,\\
&\frac{\partial Bh_n}{\partial x}=-\rho_n Ah_n, \quad \frac{\partial Bh_n}{\partial y}=\sigma_n Dh_n, \quad \frac{\partial Bh_n}{\partial z}=\mp r_n Bh_n, \quad \frac{\partial Bh_n}{\partial t}=Cx_n\rho_n Ah_n-Cy_n\sigma_n Dh_n,\\
&\frac{\partial Ch_n}{\partial x}=\rho_n Dh_n, \quad \frac{\partial Ch_n}{\partial y}=-\sigma_n Ah_n, \quad \frac{\partial Ch_n}{\partial z}=\mp r_n Ch_n, \quad \frac{\partial Ch_n}{\partial t}=-Cx_n\rho_n Dh_n+Cy_n\sigma_n Ah_n,\\
&\frac{\partial Dh_n}{\partial x}=-\rho_n Ch_n, \quad \frac{\partial Dh_n}{\partial y}=-\sigma_n Bh_n, \quad \frac{\partial Dh_n}{\partial z}=\mp r_n Dh_n, \quad \frac{\partial Dh_n}{\partial t}=Cx_n\rho_n Ch_n+Cy_n\sigma_n Bh_n.
\end{aligned} \tag{130}$$

Thus, the family of four differentially independent SKEF structures $[Ah_n, Bh_n, Ch_n, Dh_n]$ is closed with respect to the spatial and temporal differentiation of all orders.

Computing the Laplacians of (129) by (130) and identities (43) shows that the SKEF structural basis $[Ah_n, Bh_n, Ch_n, Dh_n]$ is harmonic both the upper and lower flows since

$$\begin{aligned}
&\Delta Ah_n=0, \quad \Delta Ah=0, \quad \Delta Bh_n=0, \quad \Delta Bh=0,\\
&\Delta Ch_n=0, \quad \Delta Ch=0, \quad \Delta Dh_n=0, \quad \Delta Dh=0.
\end{aligned} \tag{131}$$

By (130), dot products of gradients to local isosurfaces $Ah_n(x,y,z,t)=\text{const}$, $Bh_n(x,y,z,t)=\text{const}$, $Ch_n(x,y,z,t)=\text{const}$, $Dh_n(x,y,z,t)=\text{const}$ both for the upper and lower flows become

$$\begin{aligned}
&\frac{\partial Ah_n}{\partial x}\frac{\partial Bh_n}{\partial x}+\frac{\partial Ah_n}{\partial y}\frac{\partial Bh_n}{\partial y}+\frac{\partial Ah_n}{\partial z}\frac{\partial Bh_n}{\partial z}=\sigma_n^2\left(Ah_nBh_n+Ch_nDh_n\right),\\
&\frac{\partial Ch_n}{\partial x}\frac{\partial Dh_n}{\partial x}+\frac{\partial Ch_n}{\partial y}\frac{\partial Dh_n}{\partial y}+\frac{\partial Ch_n}{\partial z}\frac{\partial Dh_n}{\partial z}=\sigma_n^2\left(Ah_nBh_n+Ch_nDh_n\right),\\
&\frac{\partial Ah_n}{\partial x}\frac{\partial Ch_n}{\partial x}+\frac{\partial Ah_n}{\partial y}\frac{\partial Ch_n}{\partial y}+\frac{\partial Ah_n}{\partial z}\frac{\partial Ch_n}{\partial z}=\rho_n^2\left(Ah_nCh_n+Bh_nDh_n\right),\\
&\frac{\partial Bh_n}{\partial x}\frac{\partial Dh_n}{\partial x}+\frac{\partial Bh_n}{\partial y}\frac{\partial Dh_n}{\partial y}+\frac{\partial Bh_n}{\partial z}\frac{\partial Dh_n}{\partial z}=\rho_n^2\left(Ah_nCh_n+Bh_nDh_n\right),\\
&\frac{\partial Ah_n}{\partial x}\frac{\partial Dh_n}{\partial x}+\frac{\partial Ah_n}{\partial y}\frac{\partial Dh_n}{\partial y}+\frac{\partial Ah_n}{\partial z}\frac{\partial Dh_n}{\partial z}=r_n^2\left(Ah_nDh_n-Bh_nCh_n\right),\\
&\frac{\partial Bh_n}{\partial x}\frac{\partial Ch_n}{\partial x}+\frac{\partial Bh_n}{\partial y}\frac{\partial Ch_n}{\partial y}+\frac{\partial Bh_n}{\partial z}\frac{\partial Ch_n}{\partial z}=-r_n^2\left(Ah_nDh_n-Bh_nCh_n\right).
\end{aligned} \tag{132}$$

Thus, there are three couples of symmetric isosurfaces with the same absolute values of the scalar products of gradients. In two dimensions, the non-orthogonal isosurfaces are reduced to orthogonal isolines [5].

For velocity components (44), the non-orthogonal harmonic SKEF structural basis is

$$\begin{aligned}
Aw_n(x,y,z,t)&=\left(Fw_ncc_n+Qw_ncs_n+Gw_nsc_n+Rw_nss_n\right)\exp(\mp r_nz),\\
Bw_n(x,y,z,t)&=\left(Gw_ncc_n+Rw_ncs_n-Fw_nsc_n-Qw_nss_n\right)\exp(\mp r_nz),\\
Cw_n(x,y,z,t)&=\left(Qw_ncc_n-Fw_ncs_n+Rw_nsc_n-Gw_nss_n\right)\exp(\mp r_nz),\\
Dw_n(x,y,z,t)&=\left(Rw_ncc_n-Gw_ncs_n-Qw_nsc_n+Fw_nss_n\right)\exp(\mp r_nz),
\end{aligned} \tag{133}$$

where Fw_n, Qw_n, Gw_n, Rw_n are boundary coefficients provided by (23) and (25). Decomposition of the velocity components in the SKEF structural basis $[Aw_n, Bw_n, Cw_n, Dw_n](x,y,z,t)$ yields for the upper and lower flows

$$u_n=\mp C_nBw_n, \quad v_n=\mp S_nCw_n, \quad w_n=Aw_n. \tag{134}$$

Since Aw_n, Bw_n, Cw_n are harmonic functions, the local and cumulative velocity components are also harmonic both for the upper and lower flows, in agreement with

$$\Delta u_n = 0, \ \ \Delta u = 0, \ \ \Delta v_n = 0, \ \ \Delta v = 0, \ \ \Delta w_n = 0, \ \ \Delta w = 0. \tag{135}$$

Taking the temporal derivatives of (44) by (93) and reducing to the SKEF structures using (47) give

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{n=1}^{N}\left(Fut_n cc_n + Qut_n cs_n + Gut_n sc_n + Rut_n ss_n\right)\exp\left(\mp r_n z\right),\\
\frac{\partial v}{\partial t} &= \sum_{n=1}^{N}\left(Fvt_n cc_n + Qvt_n cs_n + Gvt_n sc_n + Rvt_n ss_n\right)\exp\left(\mp r_n z\right),\\
\frac{\partial w}{\partial t} &= \sum_{n=1}^{N}\left(Fwt_n cc_n + Qwt_n cs_n + Gwt_n sc_n + Rwt_n ss_n\right)\exp\left(\mp r_n z\right),
\end{aligned} \tag{136}$$

where structural coefficients are

$$\begin{aligned}
&Fut_n = \mp\rho_n\left(Cx_n C_n Fw_n - Cy_n S_n Rw_n\right), \quad Qut_n = \mp\rho_n\left(Cx_n C_n Qw_n + Cy_n S_n Gw_n\right),\\
&Gut_n = \mp\rho_n\left(Cx_n C_n Gw_n + Cy_n S_n Qw_n\right), \quad Rut_n = \mp\rho_n\left(Cx_n C_n Rw_n - Cy_n S_n Fw_n\right),
\end{aligned} \tag{137}$$

$$\begin{aligned}
&Fvt_n = \mp\sigma_n\left(-Cx_n C_n Rw_n + Cy_n S_n Fw_n\right), \quad Qvt_n = \mp\sigma_n\left(Cx_n C_n Gw_n + Cy_n S_n Qw_n\right),\\
&Gvt_n = \mp\sigma_n\left(Cx_n C_n Qw_n + Cy_n S_n Gw_n\right), \quad Rvt_n = \mp\sigma_n\left(-Cx_n C_n Fw_n + Cy_n S_n Rw_n\right),
\end{aligned} \tag{138}$$

$$\begin{aligned}
&Fwt_n = r_n\left(-Cx_n C_n Gw_n - Cy_n S_n Qw_n\right), \quad Qwt_n = r_n\left(-Cx_n C_n Rw_n + Cy_n S_n Fw_n\right),\\
&Gwt_n = r_n\left(Cx_n C_n Fw_n - Cy_n S_n Rw_n\right), \quad Rwt_n = r_n\left(Cx_n C_n Qw_n + Cy_n S_n Gw_n\right).
\end{aligned} \tag{139}$$

Decompositions of the temporal derivatives of the velocity components in the SKEF structural basis $\left[Aw_n, Bw_n, Cw_n, Dw_n\right](x, y, z, t)$ for the upper and lower flows, respectively, are

$$\begin{aligned}
&\frac{\partial u_n}{\partial t} = \mp\rho_n\left(Cx_n C_n Aw_n - Cy_n S_n Dw_n\right), \quad \frac{\partial v_n}{\partial t} = \mp\sigma_n\left(-Cx_n C_n Dw_n + Cy_n S_n Aw_n\right),\\
&\frac{\partial w_n}{\partial t} = r_n\left(-Cx_n C_n Bw_n - Cy_n S_n Cw_n\right).
\end{aligned} \tag{140}$$

For kinematic potentials (79) and dynamic potentials (97) and (105), the non-orthogonal harmonic SKEF structural basis is $\left[Aw_n, Bw_n, Cw_n, Dw_n, As_n, Bs_n, Cs_n, Ds_n\right](x, y, z, t)$, where

$$\begin{aligned}
As_n(x,y,z,t) &= \left(Fs_n cc_n + Qs_n cs_n + Gs_n sc_n + Rs_n ss_n\right)\exp\left(\mp r_n z\right),\\
Bs_n(x,y,z,t) &= \left(Gs_n cc_n + Rs_n cs_n - Fs_n sc_n - Qs_n ss_n\right)\exp\left(\mp r_n z\right),\\
Cs_n(x,y,z,t) &= \left(Qs_n cc_n - Fs_n cs_n + Rs_n sc_n - Gs_n ss_n\right)\exp\left(\mp r_n z\right),\\
Ds_n(x,y,z,t) &= \left(Rs_n cc_n - Gs_n cs_n - Qs_n sc_n + Fs_n ss_n\right)\exp\left(\mp r_n z\right),
\end{aligned} \tag{141}$$

Fs_n, Qs_n, Gs_n, Rs_n are boundary coefficients of (64). Decomposing the kinematic potentials yields for the upper and lower flows, respectively,

$$\chi_n = \mp C_n Bs_n + \frac{S_n}{r_n} Cw_n, \quad \eta_n = \mp S_n Cs_n - \frac{C_n}{r_n} Bw_n, \quad \psi_n = As_n, \quad \phi_n = \mp\frac{1}{r_n} Aw_n. \tag{142}$$

Computation of elements of the SKEF structural basis $\left[Aw_n, Bw_n, Cw_n, Dw_n\right](x, y, z, t)$ through the velocity components from (134) and substitution in (142) return local relationships between the kinematic potentials and the velocity components

$$\chi_n = \mp\left(C_n Bs_n + \frac{v_n}{r_n}\right), \quad \eta_n = \mp\left(S_n Cs_n - \frac{u_n}{r_n}\right), \quad \phi_n = \mp\frac{w_n}{r_n}. \tag{143}$$

Decomposition of the dynamic potentials in the SKEF structural basis (133), (141) gives for the upper and lower flows, respectively,

$$\begin{aligned} fe_n &= \mp\rho_n\left(Cx_nC_nAs_n - Cy_nS_nDs_n\right) - S_n\left(Cx_nC_nDw_n - Cy_nS_nAw_n\right),\\ ge_n &= \mp\sigma_n\left(-Cx_nC_nDs_n + Cy_nS_nAs_n\right) - C_n\left(Cx_nC_nAw_n - Cy_nS_nDw_n\right),\\ he_n &= -r_n\left(Cx_nC_nBs_n + Cy_nS_nCs_n\right), \quad be_n = \mp\left(Cx_nC_nBw_n + Cy_nS_nCw_n\right). \end{aligned} \tag{144}$$

Computation of elements of the SKEF structural basis by (140) and substitution in (144) returns local relationships between the dynamic potentials and the temporal derivatives of the velocity components

$$\begin{aligned} fe_n &= \mp\left[\rho_n\left(Cx_nC_nAs_n - Cy_nS_nDs_n\right) + \frac{1}{r_n}\frac{\partial v_n}{\partial t}\right],\\ ge_n &= \pm\left[\sigma_n\left(Cx_nC_nDs_n - Cy_nS_nAs_n\right) + \frac{1}{r_n}\frac{\partial u_n}{\partial t}\right], \quad be_n = \pm\frac{1}{r_n}\frac{\partial w_n}{\partial t}. \end{aligned} \tag{145}$$

Since the SKEF structural basis is harmonic, there are 11 pairs of local and global functions (u_n, u), (v_n, v), (w_n, w), (χ_n, χ), (η_n, η), (ψ_n, ψ), (ϕ_n, ϕ), (fe_n, fe), (ge_n, ge), (he_n, he), (be_n, be), which are harmonic together with all their spatial and temporal derivatives.

8. Experimental and Theoretical Verification by the System of Navier-Stokes PDEs

The classical proofs of existence theorems for series solutions of PDEs, see existence theorems of [3] and references therein, include three following steps: 1) to derive formal solutions, 2) to show that PDEs are satisfied, and 3) to find conditions of convergence. For the structural solutions of this paper, the first step is implemented in Sections 2-7, the second step is the point of this section, and the third step is not required since the structural solutions are exact and decompositions in the invariant structures are truncated.

The Navier-Stokes PDEs (1) in the scalar notation become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p_t}{\partial x} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0, \tag{146}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial p_t}{\partial y} - \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 0, \tag{147}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p_t}{\partial z} - \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + g_z = 0, \tag{148}$$

Theoretical computation of the directional derivatives of (146)-(148) by (120), (94)-(96), and (117)-(118) as a superposition of three products of the SKEF structures, which is reduced to the SDEF structure, yields for the upper and lower cumulative flows, respectively,

$$\begin{aligned} (\boldsymbol{v}\cdot\nabla)u &= \frac{1}{4}\sum_{n=1}^{N}\left(Fsddu_{n,n}CsCd_{n,n} + Gsddu_{n,n}ScCd_{n,n}\right)\exp\left(\mp 2r_nz\right)\\ &+\frac{1}{4}\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\Big(Fdddu_{n,m}CdCd_{n,m} + Fdsdu_{n,m}CdCs_{n,m} + Qdddu_{n,m}CdSd_{n,m}\\ &+ Qdsdu_{n,m}CdSs_{n,m} + Fsddu_{n,m}CsCd_{n,m} + Fssdu_{n,m}CsCs_{n,m} + Qsddu_{n,m}CsSd_{n,m}\\ &+ Qssdu_{n,m}CsSs_{n,m} + Gdddu_{n,m}SdCd_{n,m} + Gdsdu_{n,m}SdCs_{n,m} + Rdddu_{n,m}SdSd_{n,m}\\ &+ Rdsdu_{n,m}SdSs_{n,m} + Gsddu_{n,m}SsCd_{n,m} + Gssdu_{n,m}SsCs_{n,m}\\ &+ Rsddu_{n,m}SsSd_{n,m} + Rssdu_{n,m}SsSs_{n,m}\Big)\exp\left[\mp\left(r_n + r_m\right)z\right], \end{aligned} \tag{149}$$

$$\begin{aligned}(\boldsymbol{v}\cdot\nabla)v=&\frac{1}{4}\sum_{n=1}^{N}\left(Fdsdv_{n,n}CdCs_{n,n}+Qdsdv_{n,n}CdSs_{n,n}\right)\exp\left(\mp 2r_n z\right)\\&+\frac{1}{4}\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\left(Fdddv_{n,m}CdCd_{n,m}+Fdsdv_{n,m}CdCs_{n,m}+Qdddv_{n,m}CdSd_{n,m}\right.\\&+Qdsdv_{n,m}CdSs_{n,m}+Fsddv_{n,m}CsCd_{n,m}+Fssdv_{n,m}CsCs_{n,m}+Qsddv_{n,m}CsSd_{n,m}\\&+Qssdv_{n,m}CsSs_{n,m}+Gdddv_{n,m}SdCd_{n,m}+Gdsdv_{n,m}SdCs_{n,m}+Rdddv_{n,m}SdSd_{n,m}\\&+Rdsdv_{n,m}SdSs_{n,m}+Gsddv_{n,m}SsCd_{n,m}+Gssdv_{n,m}SsCs_{n,m}\\&\left.+Rsddv_{n,m}SsSd_{n,m}+Rssdv_{n,m}SsSs_{n,m}\right)\exp\left[\mp\left(r_n+r_m\right)z\right],\end{aligned}\tag{150}$$

$$\begin{aligned}(\boldsymbol{v}\cdot\nabla)w=&\frac{1}{4}\sum_{n=1}^{N}\left(Fdddw_{n,n}+Fdsdw_{n,n}CdCs_{n,n}+Qdsdw_{n,n}CdSs_{n,n}\right.\\&\left.+Fsddw_{n,n}CsCd_{n,n}+Gsddw_{n,n}ScCd_{n,n}\right)\exp\left(\mp 2r_n z\right)\\&+\frac{1}{4}\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\left(Fdddw_{n,m}CdCd_{n,m}+Fdsdw_{n,m}CdCs_{n,m}+Qdddw_{n,m}CdSd_{n,m}\right.\\&+Qdsdw_{n,m}CdSs_{n,m}+Fsddw_{n,m}CsCd_{n,m}+Fssdw_{n,m}CsCs_{n,m}+Qsddw_{n,m}CsSd_{n,m}\\&+Qssdw_{n,m}CsSs_{n,m}+Gdddw_{n,m}SdCd_{n,m}+Gdsdw_{n,m}SdCs_{n,m}+Rdddw_{n,m}SdSd_{n,m}\\&+Rdsdw_{n,m}SdSs_{n,m}+Gsddw_{n,m}SsCd_{n,m}+Gssdw_{n,m}SsCs_{n,m}\\&\left.+Rsddw_{n,m}SsSd_{n,m}+Rssdw_{n,m}SsSs_{n,m}\right)\exp\left[\mp\left(r_n+r_m\right)z\right],\end{aligned}\tag{151}$$

where the structural coefficients for Equation (149) are

$$\begin{aligned}&Fsddu_{n,n}=4\rho_n S_n^2\left(Fw_n Gw_n+Qw_n Rw_n\right),\quad Gsddu_{n,n}=-2\rho_n S_n^2\left(Fw_n^2+Qw_n^2-Gw_n^2-Rw_n^2\right),\\&Fdddu_{n,m}=\left(\rho_n-\rho_m\right)\left(1+Td_{n,m}\right)\left(-Fw_n Gw_m+Fw_m Gw_n-Qw_n Rw_m+Qw_m Rw_n\right),\\&Fdsdu_{n,m}=\left(\rho_n-\rho_m\right)\left(1+Ts_{n,m}\right)\left(-Fw_n Gw_m+Fw_m Gw_n+Qw_n Rw_m-Qw_m Rw_n\right),\\&Qdddu_{n,m}=\left(\rho_n-\rho_m\right)\left(1+Td_{n,m}\right)\left(Fw_n Rw_m+Fw_m Rw_n-Qw_n Gw_m-Qw_m Gw_n\right),\\&Qdsdu_{n,m}=\left(\rho_n-\rho_m\right)\left(1+Ts_{n,m}\right)\left(-Fw_n Rw_m+Fw_m Rw_n-Qw_n Gw_m+Qw_m Gw_n\right),\\&Fsddu_{n,m}=\left(\rho_n+\rho_m\right)\left(1-Ts_{n,m}\right)\left(Fw_n Gw_m+Fw_m Gw_n+Qw_n Rw_m+Qw_m Rw_n\right),\\&Fssdu_{n,m}=\left(\rho_n+\rho_m\right)\left(1-Td_{n,m}\right)\left(Fw_n Gw_m+Fw_m Gw_n-Qw_n Rw_m-Qw_m Rw_n\right),\\&Qsddu_{n,m}=\left(\rho_n+\rho_m\right)\left(1-Ts_{n,m}\right)\left(-Fw_n Rw_m+Fw_m Rw_n+Qw_n Gw_m-Qw_m Gw_n\right),\\&Qssdu_{n,m}=\left(\rho_n+\rho_m\right)\left(1-Td_{n,m}\right)\left(Fw_n Rw_m+Fw_m Rw_n+Qw_n Gw_m+Qw_m Gw_n\right),\\&Gdddu_{n,m}=-\left(\rho_n-\rho_m\right)\left(1+Td_{n,m}\right)\left(Fw_n Fw_m+Qw_n Qw_m+Gw_n Gw_m+Rw_n Rw_m\right),\\&Gdsdu_{n,m}=-\left(\rho_n-\rho_m\right)\left(1+Ts_{n,m}\right)\left(Fw_n Fw_m-Qw_n Qw_m+Gw_n Gw_m-Rw_n Rw_m\right),\\&Rdddu_{n,m}=-\left(\rho_n-\rho_m\right)\left(1+Td_{n,m}\right)\left(-Fw_n Qw_m+Fw_m Qw_n-Gw_n Rw_m+Gw_m Rw_n\right),\\&Rdsdu_{n,m}=-\left(\rho_n-\rho_m\right)\left(1+Ts_{n,m}\right)\left(Fw_n Qw_m+Fw_m Qw_n+Gw_n Rw_m+Gw_m Rw_n\right),\\&Gsddu_{n,m}=-\left(\rho_n+\rho_m\right)\left(1-Ts_{n,m}\right)\left(Fw_n Fw_m+Qw_n Qw_m-Gw_n Gw_m-Rw_n Rw_m\right),\\&Gssdu_{n,m}=-\left(\rho_n+\rho_m\right)\left(1-Td_{n,m}\right)\left(Fw_n Fw_m-Qw_n Qw_m-Gw_n Gw_m+Rw_n Rw_m\right),\\&Rsddu_{n,m}=-\left(\rho_n+\rho_m\right)\left(1-Ts_{n,m}\right)\left(-Fw_n Qw_m+Fw_m Qw_n+Gw_n Rw_m-Gw_m Rw_n\right),\\&Rssdu_{n,m}=-\left(\rho_n+\rho_m\right)\left(1-Td_{n,m}\right)\left(Fw_n Qw_m+Fw_m Qw_n-Gw_n Rw_m-Gw_m Rw_n\right),\end{aligned}\tag{152}$$

for Equation (150) are

$$\begin{aligned}
&Fdsdv_{n,n} = 4\sigma_n C_n^2 \left(Fw_n Qw_n + Gw_n Rw_n \right), \quad Qdsdv_{n,n} = -2\sigma_n C_n^2 \left(Fw_n^2 - Qw_n^2 + Gw_n^2 - Rw_n^2 \right),\\
&Fdddv_{n,m} = (\sigma_n - \sigma_m)(1 + Td_{n,m})(-Fw_n Qw_m + Fw_m Qw_n - Gw_n Rw_m + Gw_m Rw_n),\\
&Fdsdv_{n,m} = (\sigma_n + \sigma_m)(1 + Ts_{n,m})(Fw_n Qw_m + Fw_m Qw_n + Gw_n Rw_m + Gw_m Rw_n),\\
&Qdddv_{n,m} = -(\sigma_n - \sigma_m)(1 + Td_{n,m})(Fw_n Fw_m + Qw_n Qw_m + Gw_n Gw_m + Rw_n Rw_m),\\
&Qdsdv_{n,m} = -(\sigma_n + \sigma_m)(1 + Ts_{n,m})(Fw_n Fw_m - Qw_n Qw_m + Gw_n Gw_m - Rw_n Rw_m),\\
&Fsddv_{n,m} = (\sigma_n - \sigma_m)(1 - Ts_{n,m})(-Fw_n Qw_m + Fw_m Qw_n + Gw_n Rw_m - Gw_m Rw_n),\\
&Fssdv_{n,m} = (\sigma_n + \sigma_m)(1 - Td_{n,m})(Fw_n Qw_m + Fw_m Qw_n - Gw_n Rw_m - Gw_m Rw_n),\\
&Qsddv_{n,m} = -(\sigma_n - \sigma_m)(1 - Ts_{n,m})(Fw_n Fw_m + Qw_n Qw_m - Gw_n Gw_m - Rw_n Rw_m),\\
&Qssdv_{n,m} = -(\sigma_n + \sigma_m)(1 - Td_{n,m})(Fw_n Fw_m - Qw_n Qw_m - Gw_n Gw_m + Rw_n Rw_m),\\
&Gdddv_{n,m} = (\sigma_n - \sigma_m)(1 + Td_{n,m})(Fw_n Rw_m + Fw_m Rw_n - Qw_n Gw_m - Qw_m Gw_n),\\
&Gdsdv_{n,m} = (\sigma_n + \sigma_m)(1 + Ts_{n,m})(-Fw_n Rw_m + Fw_m Rw_n - Qw_n Gw_m + Qw_m Gw_n),\\
&Rdddv_{n,m} = -(\sigma_n - \sigma_m)(1 + Td_{n,m})(-Fw_n Gw_m + Fw_m Gw_n - Qw_n Rw_m + Qw_m Rw_n),\\
&Rdsdv_{n,m} = -(\sigma_n + \sigma_m)(1 + Ts_{n,m})(-Fw_n Gw_m + Fw_m Gw_n + Qw_n Rw_m - Qw_m Rw_n),\\
&Gsddv_{n,m} = (\sigma_n - \sigma_m)(1 - Ts_{n,m})(-Fw_n Rw_m + Fw_m Rw_n + Qw_n Gw_m - Qw_m Gw_n),\\
&Gssdv_{n,m} = (\sigma_n + \sigma_m)(1 - Td_{n,m})(Fw_n Rw_m + Fw_m Rw_n + Qw_n Gw_m + Qw_m Gw_n),\\
&Rsddv_{n,m} = -(\sigma_n - \sigma_m)(1 - Ts_{n,m})(Fw_n Gw_m + Fw_m Gw_n + Qw_n Rw_m + Qw_m Rw_n),\\
&Rssdv_{n,m} = -(\sigma_n + \sigma_m)(1 - Td_{n,m})(Fw_n Gw_m + Fw_m Gw_n - Qw_n Rw_m - Qw_m Rw_n),
\end{aligned} \tag{153}$$

and for Equation (151) are

$$\begin{aligned}
&Fdddw_{n,n} = \mp 2r_n \left(Fw_n^2 + Qw_n^2 + Gw_n^2 + Rw_n^2 \right),\\
&Fdsdw_{n,n} = \mp 2r_n C_n^2 \left(Fw_n^2 - Qw_n^2 + Gw_n^2 - Rw_n^2 \right), \quad Qdsdw_{n,n} = \mp 4r_n C_n^2 \left(Fw_n Qw_n + Gw_n Rw_n \right),\\
&Fsddw_{n,n} = \mp 2r_n S_n^2 \left(Fw_n^2 + Qw_n^2 - Gw_n^2 - Rw_n^2 \right), \quad Gsddw_{n,n} = \mp 4r_n S_n^2 \left(Fw_n Gw_n + Qw_n Rw_n \right),\\
&Fdddw_{n,m} = \mp (r_n + r_m)(1 + Td_{n,m})(Fw_n Fw_m + Qw_n Qw_m + Gw_n Gw_m + Rw_n Rw_m),\\
&Fdsdw_{n,m} = \mp (r_n + r_m)(1 + Ts_{n,m})(Fw_n Fw_m - Qw_n Qw_m + Gw_n Gw_m - Rw_n Rw_m),\\
&Qdddw_{n,m} = \mp (r_n + r_m)(1 + Td_{n,m})(-Fw_n Qw_m + Fw_m Qw_n - Gw_n Rw_m + Gw_m Rw_n),\\
&Qdsdw_{n,m} = \mp (r_n + r_m)(1 + Ts_{n,m})(Fw_n Qw_m + Fw_m Qw_n + Gw_n Rw_m + Gw_m Rw_n),\\
&Fsddw_{n,m} = \mp (r_n + r_m)(1 - Ts_{n,m})(Fw_n Fw_m + Qw_n Qw_m - Gw_n Gw_m - Rw_n Rw_m),\\
&Fssdw_{n,m} = \mp (r_n + r_m)(1 - Td_{n,m})(Fw_n Fw_m - Qw_n Qw_m - Gw_n Gw_m + Rw_n Rw_m),\\
&Qsddw_{n,m} = \mp (r_n + r_m)(1 - Ts_{n,m})(-Fw_n Qw_m + Fw_m Qw_n + Gw_n Rw_m - Gw_m Rw_n),\\
&Qssdw_{n,m} = \mp (r_n + r_m)(1 - Td_{n,m})(Fw_n Qw_m + Fw_m Qw_n - Gw_n Rw_m - Gw_m Rw_n),\\
&Gdddw_{n,m} = \mp (r_n + r_m)(1 + Td_{n,m})(-Fw_n Gw_m + Fw_m Gw_n - Qw_n Rw_m + Qw_m Rw_n),\\
&Gdsdw_{n,m} = \mp (r_n + r_m)(1 + Ts_{n,m})(-Fw_n Gw_m + Fw_m Gw_n + Qw_n Rw_m - Qw_m Rw_n),\\
&Rdddw_{n,m} = \mp (r_n + r_m)(1 + Td_{n,m})(Fw_n Rw_m + Fw_m Rw_n - Qw_n Gw_m - Qw_m Gw_n),\\
&Rdsdw_{n,m} = \mp (r_n + r_m)(1 + Ts_{n,m})(-Fw_n Rw_m + Fw_m Rw_n - Qw_n Gw_m + Qw_m Gw_n),\\
&Gsddw_{n,m} = \mp (r_n + r_m)(1 - Ts_{n,m})(Fw_n Gw_m + Fw_m Gw_n + Qw_n Rw_m + Qw_m Rw_n),\\
&Gssdw_{n,m} = \mp (r_n + r_m)(1 - Td_{n,m})(Fw_n Gw_m + Fw_m Gw_n - Qw_n Rw_m - Qw_m Rw_n),\\
&Rsddw_{n,m} = \mp (r_n + r_m)(1 - Ts_{n,m})(-Fw_n Rw_m + Fw_m Rw_n + Qw_n Gw_m - Qw_m Gw_n),\\
&Rssdw_{n,m} = \mp (r_n + r_m)(1 - Td_{n,m})(Fw_n Rw_m + Fw_m Rw_n + Qw_n Gw_m + Qw_m Gw_n).
\end{aligned} \tag{154}$$

In the vector description, the pressure force of (146)-(148) is reduced by (5) and (125) to

$$\frac{1}{\rho}\frac{\partial p_t}{\partial x}=\frac{\partial be}{\partial x}-\frac{\partial k_e}{\partial x},\quad \frac{1}{\rho}\frac{\partial p_t}{\partial y}=\frac{\partial be}{\partial y}-\frac{\partial k_e}{\partial y},\quad \frac{1}{\rho}\frac{\partial p_t}{\partial z}=\frac{\partial be}{\partial z}-\frac{\partial k_e}{\partial z}-g_z. \tag{155}$$

Theoretical computation of the SKEF structures for the spatial derivatives of the Bernoulli potential (105)-(106) by (94)-(96) and comparison with the SKEF structures for the temporal derivatives of the velocity components (136)-(139) yields that these structures are related by

$$\frac{\partial be}{\partial x}+\frac{\partial u}{\partial t}=0,\quad \frac{\partial be}{\partial y}+\frac{\partial v}{\partial t}=0,\quad \frac{\partial be}{\partial z}+\frac{\partial w}{\partial t}=0, \tag{156}$$

in agreement with decompositions in the SKEF structural basis of Section 7.

Taking theoretically the spatial derivative of the SDEF structure (117) in the x-direction gives for the upper and lower cumulative flows, respectively,

$$\begin{aligned}\frac{\partial p}{\partial x}=&\frac{1}{4}\sum_{n=1}^{N}\big(Fsdpx_{n,n}CsCd_{n,n}+Fsspx_{n,n}CsCs_{n,n}+Qsspx_{n,n}CsSs_{n,n}\\&+Gsdpx_{n,n}ScCd_{n,n}+Gsspx_{n,n}ScCs_{n,n}+Rsspx_{n,n}SsSs_{n,n}\big)\exp\left(\mp 2r_n z\right)\\&+\frac{1}{4}\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\big(Fddpx_{n,m}CdCd_{n,m}+Fdspx_{n,m}CdCs_{n,m}+Qddpx_{n,m}CdSd_{n,m}\\&+Qdspx_{n,m}CdSs_{n,m}+Fsdpx_{n,m}CsCd_{n,m}+Fsspx_{n,m}CsCs_{n,m}+Qsdpx_{n,m}CsSd_{n,m}\\&+Qsspx_{n,m}CsSs_{n,m}+Gddpx_{n,m}SdCd_{n,m}+Gdspx_{n,m}SdCs_{n,m}+Rddpx_{n,m}SdSd_{n,m}\\&+Rdspx_{n,m}SdSs_{n,m}+Gsdpx_{n,m}SsCd_{n,m}+Gsspx_{n,m}SsCs_{n,m}\\&+Rsdpx_{n,m}SsSd_{n,m}+Rsspx_{n,m}SsSs_{n,m}\big)\exp\left[\mp\left(r_n+r_m\right)z\right],\end{aligned} \tag{157}$$

where the structural coefficients are

$$\begin{aligned}&Fsdpx_{n,n}=2\rho_n Gsdp_{n,n},\quad Fsspx_{n,n}=2\rho_n Gssp_{n,n},\quad Qsspx_{n,n}=2\rho_n Rssp_{n,n},\\&Gsdpx_{n,n}=-2\rho_n Fsdp_{n,n},\quad Gsspx_{n,n}=-2\rho_n Fssp_{n,n},\quad Rsspx_{n,n}=-2\rho_n Qssp_{n,n},\\&Fddpx_{n,m}=\left(\rho_n-\rho_m\right)Gddp_{n,m},\quad Fdspx_{n,m}=\left(\rho_n-\rho_m\right)Gdsp_{n,m},\\&Qddpx_{n,m}=\left(\rho_n-\rho_m\right)Rddp_{n,m},\quad Qdspx_{n,m}=\left(\rho_n-\rho_m\right)Rdsp_{n,m},\\&Fsdpx_{n,m}=\left(\rho_n+\rho_m\right)Gsdp_{n,m},\quad Fsspx_{n,m}=\left(\rho_n+\rho_m\right)Gssp_{n,m},\\&Qsdpx_{n,m}=\left(\rho_n+\rho_m\right)Rsdp_{n,m},\quad Qsspx_{n,m}=\left(\rho_n+\rho_m\right)Rssp_{n,m},\\&Gddpx_{n,m}=-\left(\rho_n-\rho_m\right)Fddp_{n,m},\quad Gdspx_{n,m}=-\left(\rho_n-\rho_m\right)Fdsp_{n,m},\\&Rddpx_{n,m}=-\left(\rho_n-\rho_m\right)Qddp_{n,m},\quad Rdspx_{n,m}=-\left(\rho_n-\rho_m\right)Qdsp_{n,m},\\&Gsdpx_{n,m}=-\left(\rho_n+\rho_m\right)Fsdp_{n,m},\quad Gsspx_{n,m}=-\left(\rho_n+\rho_m\right)Fssp_{n,m},\\&Rsdpx_{n,m}=-\left(\rho_n+\rho_m\right)Qsdp_{n,m},\quad Rsspx_{n,m}=-\left(\rho_n+\rho_m\right)Qssp_{n,m}.\end{aligned} \tag{158}$$

Similarly, the spatial derivative of the SDEF structure (117) with respect to y becomes

$$\begin{aligned}\frac{\partial p}{\partial y}=&\frac{1}{4}\sum_{n=1}^{N}\big(Fdspy_{n,n}CdCs_{n,n}+Qdspy_{n,n}CdSs_{n,n}+Fsspy_{n,n}CsCs_{n,n}\\&+Qsspy_{n,n}CsSs_{n,n}+Gsspy_{n,n}ScCs_{n,n}+Rsspy_{n,n}SsSs_{n,n}\big)\exp\left(\mp 2r_n z\right)\\&+\frac{1}{4}\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\big(Fddpy_{n,m}CdCd_{n,m}+Fdspy_{n,m}CdCs_{n,m}+Qddpy_{n,m}CdSd_{n,m}\\&+Qdspy_{n,m}CdSs_{n,m}+Fsdpy_{n,m}CsCd_{n,m}+Fsspy_{n,m}CsCs_{n,m}+Qsdpy_{n,m}CsSd_{n,m}\\&+Qsspy_{n,m}CsSs_{n,m}+Gddpy_{n,m}SdCd_{n,m}+Gdspy_{n,m}SdCs_{n,m}+Rddpy_{n,m}SdSd_{n,m}\\&+Rdspy_{n,m}SdSs_{n,m}+Gsdpy_{n,m}SsCd_{n,m}+Gsspy_{n,m}SsCs_{n,m}\\&+Rsdpy_{n,m}SsSd_{n,m}+Rsspy_{n,m}SsSs_{n,m}\big)\exp\left[\mp\left(r_n+r_m\right)z\right],\end{aligned} \tag{159}$$

where the structural coefficients are

$$\begin{aligned}
&Fdspy_{n,n} = 2\sigma_n Qdsp_{n,n}, \quad Qdspy_{n,n} = -2\sigma_n Fdsp_{n,n}, \quad Fsspy_{n,n} = 2\sigma_n Qssp_{n,n},\\
&Qsspy_{n,n} = -2\sigma_n Fssp_{n,n}, \quad Gsspy_{n,n} = 2\sigma_n Rssp_{n,n}, \quad Rsspy_{n,n} = -2\sigma_n Gssp_{n,n},\\
&Fddpy_{n,m} = (\sigma_n - \sigma_m) Qddp_{n,m}, \quad Fdspy_{n,m} = (\sigma_n + \sigma_m) Qdsp_{n,m},\\
&Qddpy_{n,m} = -(\sigma_n - \sigma_m) Fddp_{n,m}, \quad Qdspy_{n,m} = -(\sigma_n + \sigma_m) Fdsp_{n,m},\\
&Fsdpy_{n,m} = (\sigma_n - \sigma_m) Qsdp_{n,m}, \quad Fsspy_{n,m} = (\sigma_n + \sigma_m) Qssp_{n,m},\\
&Qsdpy_{n,m} = -(\sigma_n - \sigma_m) Fsdp_{n,m}, \quad Qsspy_{n,m} = -(\sigma_n + \sigma_m) Fssp_{n,m},\\
&Gddpy_{n,m} = (\sigma_n - \sigma_m) Rddp_{n,m}, \quad Gdspy_{n,m} = (\sigma_n + \sigma_m) Rdsp_{n,m},\\
&Rddpy_{n,m} = -(\sigma_n - \sigma_m) Gddp_{n,m}, \quad Rdspy_{n,m} = -(\sigma_n + \sigma_m) Gdsp_{n,m},\\
&Gsdpy_{n,m} = (\sigma_n - \sigma_m) Rsdp_{n,m}, \quad Gsspy_{n,m} = (\sigma_n + \sigma_m) Rssp_{n,m},\\
&Rsdpy_{n,m} = -(\sigma_n - \sigma_m) Gsdp_{n,m}, \quad Rsspy_{n,m} = -(\sigma_n + \sigma_m) Gssp_{n,m}.
\end{aligned} \tag{160}$$

Finally, theoretical computation of the spatial derivative of the SDEF structure (117) in the z-direction returns for the upper and lower cumulative flows, respectively,

$$\begin{aligned}
\frac{\partial p}{\partial z} = &\frac{1}{4}\sum_{n=1}^{N}\big(Fddpz_{n,n} + Fdspz_{n,n} CdCs_{n,n} + Qdspz_{n,n} CdSs_{n,n} + Fsdpz_{n,n} CsCd_{n,n} + Fsspz_{n,n} CsCs_{n,n}\\
&+ Qsspz_{n,n} CsSs_{n,n} + Gsdpz_{n,n} ScCd_{n,n} + Gsspz_{n,n} ScCs_{n,n} + Rsspz_{n,n} SsSs_{n,n}\big)\exp(\mp 2r_n z)\\
&+\frac{1}{4}\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\big(Fddpz_{n,m} CdCd_{n,m} + Fdspz_{n,m} CdCs_{n,m} + Qddpz_{n,m} CdSd_{n,m}\\
&+ Qdspz_{n,m} CdSs_{n,m} + Fsdpz_{n,m} CsCd_{n,m} + Fsspz_{n,m} CsCs_{n,m} + Qsdpz_{n,m} CsSd_{n,m}\\
&+ Qsspz_{n,m} CsSs_{n,m} + Gddpz_{n,m} SdCd_{n,m} + Gdspz_{n,m} SdCs_{n,m} + Rddpz_{n,m} SdSd_{n,m}\\
&+ Rdspz_{n,m} SdSs_{n,m} + Gsdpz_{n,m} SsCd_{n,m} + Gsspz_{n,m} SsCs_{n,m}\\
&+ Rsdpz_{n,m} SsSd_{n,m} + Rsspz_{n,m} SsSs_{n,m}\big)\exp\left[\mp(r_n + r_m) z\right],
\end{aligned} \tag{161}$$

where the structural coefficients are

$$\begin{aligned}
&Fddpz_{n,n} = \mp 2r_n Fddp_{n,n}, \quad Fdspz_{n,n} = \mp 2r_n Fdsp_{n,n}, \quad Qdspz_{n,n} = \pm 2r_n Qdsp_{n,n},\\
&Fsdpz_{n,n} = \mp 2r_n Fsdp_{n,n}, \quad Fsspz_{n,n} = \mp 2r_n Fssp_{n,n}, \quad Qsspz_{n,n} = \pm 2r_n Qssp_{n,n},\\
&Gsdpz_{n,n} = \mp 2r_n Gsdp_{n,n}, \quad Gsspz_{n,n} = \mp 2r_n Gssp_{n,n}, \quad Rsspz_{n,n} = \mp 2r_n Rssp_{n,n},\\
&Fddpz_{n,m} = \mp(r_n + r_m) Fddp_{n,m}, \quad Fdspz_{n,m} = \mp(r_n + r_m) Fdsp_{n,m},\\
&Qddpz_{n,m} = \mp(r_n + r_m) Qddp_{n,m}, \quad Qdspz_{n,m} = \mp(r_n + r_m) Qdsp_{n,m},\\
&Fsdpz_{n,m} = \mp(r_n + r_m) Fsdp_{n,m}, \quad Fsspz_{n,m} = \mp(r_n + r_m) Fssp_{n,m},\\
&Qsdpz_{n,m} = \mp(r_n + r_m) Qsdp_{n,m}, \quad Qsspz_{n,m} = \mp(r_n + r_m) Qssp_{n,m},\\
&Gddpz_{n,m} = \mp(r_n + r_m) Gddp_{n,m}, \quad Gdspz_{n,m} = \mp(r_n + r_m) Gdsp_{n,m},\\
&Rddpz_{n,m} = \mp(r_n + r_m) Rddp_{n,m}, \quad Rdspz_{n,m} = \mp(r_n + r_m) Rdsp_{n,m},\\
&Gsdpz_{n,m} = \mp(r_n + r_m) Gsdp_{n,m}, \quad Gsspz_{n,m} = \mp(r_n + r_m) Gssp_{n,m},\\
&Rsdpz_{n,m} = \mp(r_n + r_m) Rsdp_{n,m}, \quad Rsspz_{n,m} = \mp(r_n + r_m) Rssp_{n,m}.
\end{aligned} \tag{162}$$

Thus, Equations (157)-(162) and (93)-(96) prove that the SDEF and SKEF structures are invariant with respect to spatial and temporal differentiation.

Application of (157)-(162) to the SDEF structure for the kinetic energy (122)-(123) and comparison with the SDEF structures for the directional derivatives of the velocity components (149)-(154) yields that these structures, similar to the SKEF structures in Section 7, are connected by

$$(\boldsymbol{v}\cdot\nabla)u-\frac{\partial k_e}{\partial x}=0,\quad (\boldsymbol{v}\cdot\nabla)v-\frac{\partial k_e}{\partial y}=0,\quad (\boldsymbol{v}\cdot\nabla)w-\frac{\partial k_e}{\partial z}=0. \tag{163}$$

Substitution of (135), (155), (156), and (163) in Equations (146)-(148) shows that the Navier-Stokes PDEs are satisfied exactly by the structural solutions in the SKEF, SDEF, and polynomial structures both for the upper and lower cumulative flows. Similarly, computation of the spatial derivatives of the SKEF structures (120) for the velocity components by (94)-(96) and substitution in the continuity PDE (17) also reduces it to identity for the upper and lower cumulative flows.

The main results of this paper are summarized in the following theorem.

Existence Theorem. In the C^{∞} class of the SKEF structures (92) with four structural functions $[cc_n, cs_n, sc_n, ss_n]$ defined by (27) and four constant structural coefficients $[Fl_n, Ql_n, Gl_n, Rl_n]$, the SDEF structures (117) with 24 structural functions

$$\begin{gathered}[CdCs_{n,n}, CdSs_{n,n}, CsCd_{n,n}, CsCs_{n,n}, CsSs_{n,n}, ScCd_{n,n}, ScCs_{n,n}, SsSs_{n,n}, CdCd_{n,m}, CdCs_{n,m}, CdSd_{n,m}, CdSs_{n,m},\\ CsCd_{n,m}, CsCs_{n,m}, CsSd_{n,m}, CsSs_{n,m}, SdCd_{n,m}, SdCs_{n,m}, SdSd_{n,m}, SdSs_{n,m}, SsCd_{n,m}, SsCs_{n,m}, SsSd_{n,m}, SsSs_{n,m}],\end{gathered}$$

defined by (112) and 25 constant structural coefficients

$$\begin{gathered}[Fddp_{n,n}, Fdsp_{n,n}, Qdsp_{n,n}, Fsdp_{n,n}, Fssp_{n,n}, Qssp_{n,n}, Gsdp_{n,n}, Gssp_{n,n}, Rssp_{n,n},\\ Fddp_{n,m}, Fdsp_{n,m}, Qddp_{n,m}, Qdsp_{n,m}, Fsdp_{n,m}, Fssp_{n,m}, Qsdp_{n,m}, Qssp_{n,m},\\ Gddp_{n,m}, Gdsp_{n,m}, Rddp_{n,m}, Rdsp_{n,m}, Gsdp_{n,m}, Gssp_{n,m}, Rsdp_{n,m}, Rssp_{n,m}],\end{gathered}$$

and the polynomial structures, there are upper and lower exact solutions for the velocity components $[u, v, w]$ (120)-(121) and the total pressure p_t (126), (106), (123) of the system of the Navier-Stokes PDEs (146)-(148) and (17) for conservative interaction of N internal waves in $x\in(-\infty,\infty)$, $y\in(-\infty,\infty)$, $z\in[0,\infty)$, $t\in(-\infty,\infty)$ and $x\in(-\infty,\infty)$, $y\in(-\infty,\infty)$, $z\in(0,\infty]$, $t\in(-\infty,\infty)$ respectively. The structural solutions are unique if the Dirichlet conditions (23), (25), (64), which are periodic in the *x*- and *y*-directions, are set together with the Dirichlet conditions (24), (26) vanishing at infinity in the *z*-direction. For the upper and lower cumulative flows, there are dual presentations for the velocity components through the scalar potential ϕ and the vector potential $[\chi, \eta, \psi]$ (78)-(81) that yield dual solutions for the dynamic potentials $[fe, ge, he, be]$ (97)-(100), (105)-(110), but does not affect the uniqueness of $[u, v, w]$ and p_t.

The structural solutions were computed through the theoretical programming with symbolic general terms, symbolic indices, and code-generated names of structural variables for a symbolic number of internal waves N in the virtual environment of the global variable *Eqt* by 33 developed procedures of 1938 code lines. Primarily, the exact solutions for the upper and lower cumulative flows with were discovered through the experimental programming for $N = 3$ and then verified by the system of the Navier-Stokes PDEs. Secondly, the cumulative solutions for the velocity components (44), the kinematic potentials (78)-(81), the dynamic potentials (97)-(100), (105)-106), the product of the SKEF structures (117)-(118), the total pressure (126), (106), (123), the temporal derivatives of the velocity components (136)-(139), the directional derivatives of the velocity components (149)-(154), the spatial derivatives of the Bernoulli potential (156), and the spatial derivatives (157)-(162) of the kinetic energy (122)-(123) of the upper and lower cumulative flows were derived through the theoretical programming and justified by the correspondent experimental solutions for $N = 1, 3, 10$. Finally, the theoretical solutions for the general terms of the spatial (94)-(96), temporal (136)-(139) and directional derivatives (149)-(154) of the velocity components (120)-(121), the spatial derivatives of the Bernoulli potential (105)-(106), and the spatial derivatives (157)-(162) of the kinetic energy (122)-(123) were verified by the system of the Navier-Stokes PDEs.

9. Discussion and Visualization

The structural solutions for conservative interaction of N internal waves in the upper and lower domains model generation, propagation, and interaction of internal waves in the atmosphere and ocean. For fluid-dynamic engineering, existence of these exact solutions of the Navier-Stokes PDEs means existence of an enormous source of the kinetic energy of the internal waves beneath the ocean surface. This source of energy is continuously maintained by perpetual surface waves and it is not affected by viscous dissipation since the velocity field is harmonic.

The exact structural solutions are neutrally stable with respect to any number M of wave perturbations in the class of SKEF structures since the resulting flows are reduced to the upper and lower cumulative flows with $N + M$ internal waves. Since the upper and lower cumulative flows are harmonic, the structural solutions do not depend on the Reynolds number, similar to how all solutions for conservative systems do not depend on dissipation parameters. For the same reason, initial conditions are not required for conservative PDEs, which produce solutions propagating for all times. In conservative solutions, any moment may be treated as an initial moment.

The implemented experimental and theoretical programming methods represent an indispensable Computational Mathematics (COMP MATH) tool, without which discovery and proof of the structural solutions would be impossible since the artificial intelligence of the Maple theoretical computation programs far exceeds the human intelligence. For instance, a calculation of the directional derivative of a single scalar variable u, which produces Equations (149) and (152) for the upper cumulative flow, requires 100 substitutions for 100 structural coefficients with the code-generated names:

$$\begin{aligned}&\big[Fdddu_{n,n}, Fdsdu_{n,n}, Qdsdu_{n,n}, Fsddu_{n,n}, Fssdu_{n,n}, Qssdu_{n,n}, Gsddu_{n,n}, Gssdu_{n,n},\\ &Rssdu_{n,n}, Fdddu_{n,m}, Fdsdu_{n,m}, Qdddu_{n,m}, Qdsdu_{n,m}, Fsddu_{n,m}, Fssdu_{n,m}, Qsddu_{n,m},\\ &Qssdu_{n,m}, Gdddu_{n,m}, Gdsdu_{n,m}, Rdddu_{n,m}, Rdsdu_{n,m}, Gsddu_{n,m}, Gssdu_{n,m}, Rsddu_{n,m},\\ &Rssdu_{n,m}, Fdddux_{n,n}, Fdsdux_{n,n}, Qdsdux_{n,n}, Fsddux_{n,n}, Fssdux_{n,n}, Qssdux_{n,n},\\ &Gsddux_{n,n}, Gssdux_{n,n}, Rssdux_{n,n}, Fdddux_{n,m}, Fdsdux_{n,m}, Qdddux_{n,m}, Qdsdux_{n,m},\\ &Fsddux_{n,m}, Fssdux_{n,m}, Qsddux_{n,m}, Qssdux_{n,m}, Gdddux_{n,m}, Gdsdux_{n,m}, Rdddux_{n,m},\\ &Rdsdux_{n,m}, Gsddux_{n,m}, Gssdux_{n,m}, Rsddux_{n,m}, Rssdux_{n,m}, Fddduy_{n,n}, Fdsduy_{n,n},\\ &Qdsduy_{n,n}, Fsdduy_{n,n}, Fssduy_{n,n}, Qssduy_{n,n}, Gsdduy_{n,n}, Gssduy_{n,n}, Rssduy_{n,n},\\ &Fddduy_{n,m}, Fdsduy_{n,m}, Qddduy_{n,m}, Qdsduy_{n,m}, Fsdduy_{n,m}, Fssduy_{n,m}, Qsdduy_{n,m},\\ &Qssduy_{n,m}, Gddduy_{n,m}, Gdsduy_{n,m}, Rddduy_{n,m}, Rdsduy_{n,m}, Gsdduy_{n,m}, Gssduy_{n,m},\\ &Rsdduy_{n,m}, Rssduy_{n,m}, Fddduz_{n,n}, Fdsduz_{n,n}, Qdsduz_{n,n}, Fsdduz_{n,n}, Fssduz_{n,n},\\ &Qssduz_{n,n}, Gsdduz_{n,n}, Gssduz_{n,n}, Rssduz_{n,n}, Fddduz_{n,m}, Fdsduz_{n,m}, Qddduz_{n,m},\\ &Qdsduz_{n,m}, Fsdduz_{n,m}, Fssduz_{n,m}, Qsdduz_{n,m}, Qssduz_{n,m}, Gddduz_{n,m}, Gdsduz_{n,m},\\ &Rddduz_{n,m}, Rdsduz_{n,m}, Gsdduz_{n,m}, Gssduz_{n,m}, Rsdduz_{n,m}, Rssduz_{n,m}\big].\end{aligned} \tag{164}$$

The listed names of computational variables in fact are words of a new computational language, which is required to compute theoretically exact formulas for the new COMP MATH data structures: the SKEF and SDEF structures, while this COMP MATH language is developed in the process of computation by the Maple program itself.

The structural solutions in the SKEF structures for the velocity components (120)-(121), the kinematic potentials (78)-(81), and the dynamic potentials (97)-(100), in the SDEF structures for the kinetic energy (122)-(123), in the SKEF-SDEF structures for the dynamic pressure (125), and in the SKEF-SDEF and polynomial structures for the total pressure (126), (106), (123) depend on 14N independent parameters $\big[\rho_n, \sigma_n, Cx_n, Cy_n, Xa_n, Yb_n, Fw_n, Qw_n, Gw_n, Rw_n, Fs_n, Qs_n, Gs_n, Rs_n\big]$. The trigonometric structural functions $\big[cc_n, cs_n, sc_n, ss_n\big]$ (27) of the SKEF structures are doubly periodic functions and coincide with the 2d Fourier eigenfunctions when $\rho_n = n_x\lambda_x$ and $\sigma_n = n_y\lambda_y$, here $\big[n_x, n_y\big]$ are integers, $\big[\lambda_x = 2\pi/P_x, \lambda_y = 2\pi/P_y\big]$ are unit wave numbers, and $\big[P_x, P_y\big]$ are periods. When $\rho_n = \lambda_x/q_x$ and $\sigma_n = \lambda_y/q_y$, where $\big[q_x, q_y\big]$ are primes, $\big[cc_n, cs_n, sc_n, ss_n\big]$ model doubly periodic functions with a period approaching infinity as $n \to \infty$ [5] [10].

The SKEF structures of the lower cumulative flow are visualized for the scalar potential ϕ (79), (81) and three components of the vector potential $\big[\chi, \eta, \psi\big]$ (78)-(80) of the velocity field $[u, v, w]$ (120), (121) in **Figure 2(a)-(d)** by instantaneous plots of two-parametric surfaces $\phi = \phi(x, y, z_0, t_0)$, $\chi = \chi(x, y, z_0, t_0)$, $\psi = \psi(x, y, z_0, t_0)$, respectively, for the following parameters:

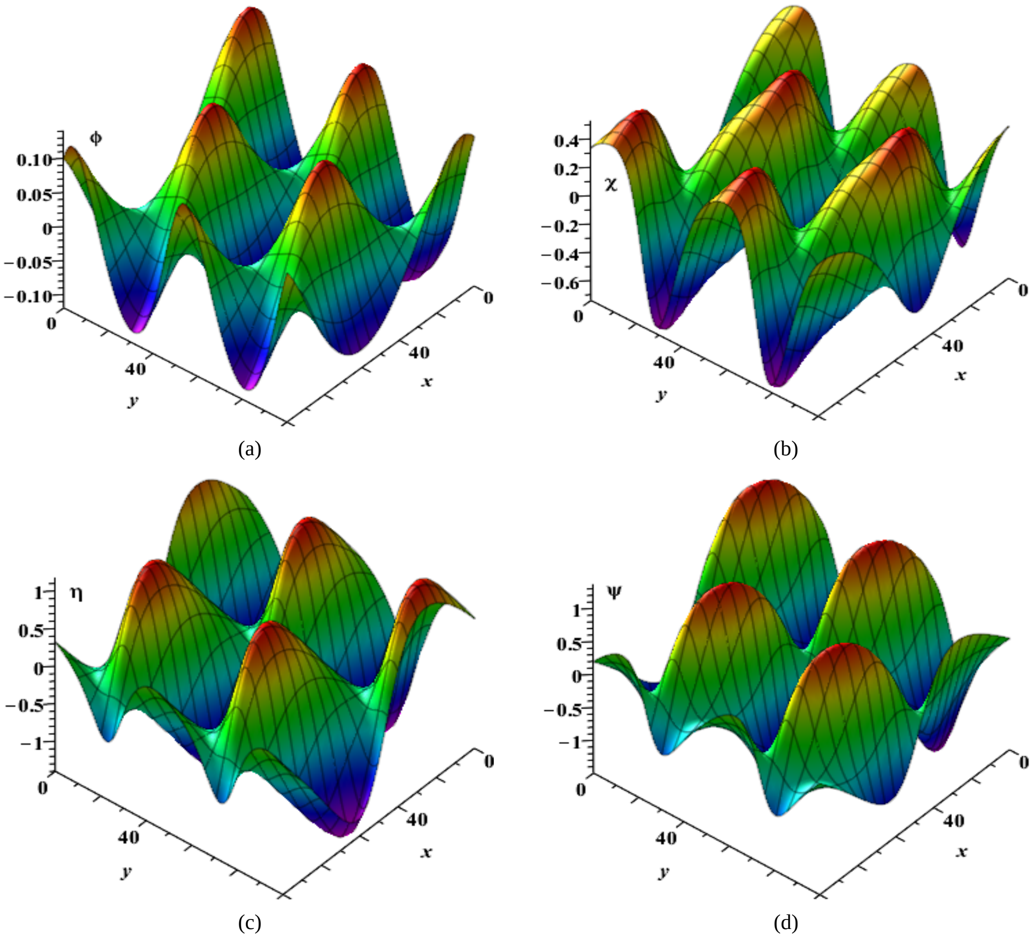

Figure 2. Kinematic potentials of the lower cumulative flow: (a) $-\phi$ (b) $-\chi$ (c) $-\eta$ (d) $-\psi$.

$$N=5,\ t_0=5,\ z_0=-0.1,\ x\in[0,100],\ y\in[0,100],\ n_x=n_y=[1,2,3,4,5],\ P_x=1,\ P_y=1/2,$$

$$Cx_n=[2,1.8,1.6,1.4,1.2],\ Cx_y=[1,0.9,0.8,0.7,0.6],\ Xa_n=[1,2,3,4,5],\ Yb_n=[2,3,4,5,6],$$

$$Fw_n=[5,4.9,4.8,4.7,4.6],\ Qw_n=[4,3.9,3.8,3.7,3.6],\ Gw_n=[3,2.9,2.8,2.7,2.6],$$

$$Rw_n=[2,1.9,1.8,1.7,1.6],\ Fs_n=[1,-1.1,1.2,-1.3,1.4],\ Qs_n=[2,-2.1,2.2,-2.3,2.4],$$

$$Gs_n=[3,-3.1,3.2,-3.3,3.4],\ Rs_n=[4,-4.1,4.2,-4.3,4.4].$$

The SDEF structures are shown for the kinetic energy k_e (122)-(123) and the dynamic pressure p_d (125), (105), (106) in **Figure 3(a)**, **Figure 3(b)** by instantaneous plots of two-parametric surfaces $k=k_e(x,y,z_0,t_0)$, $p=p_d(x,y,z_0,t_0)$, respectively, for the same parameters as in **Figure 2**. Contrary to the oscillating surfaces of the SKEF structures in **Figure 2**, the surfaces of the SDEF structures in **Figure 3** exhibit positive pulsations for k_e and negative pulsations with respect to the hydrostatic pressure combined with oscillations produced by *be* for p_d .

10. Conclusions

For the Navier-Stokes system of PDEs in three dimensions, the global existence theorem for periodic solutions vanishing at infinity of the upper and lower domains is proved by the C^∞ class of the SKEF and SDEF structures. Two exact solutions for conservative interaction of *N* internal waves are computed by formulating and solving the Dirichlet problems for the vorticity, continuity, Helmholtz, Lamb-Helmholtz, and Bernoulli equa-

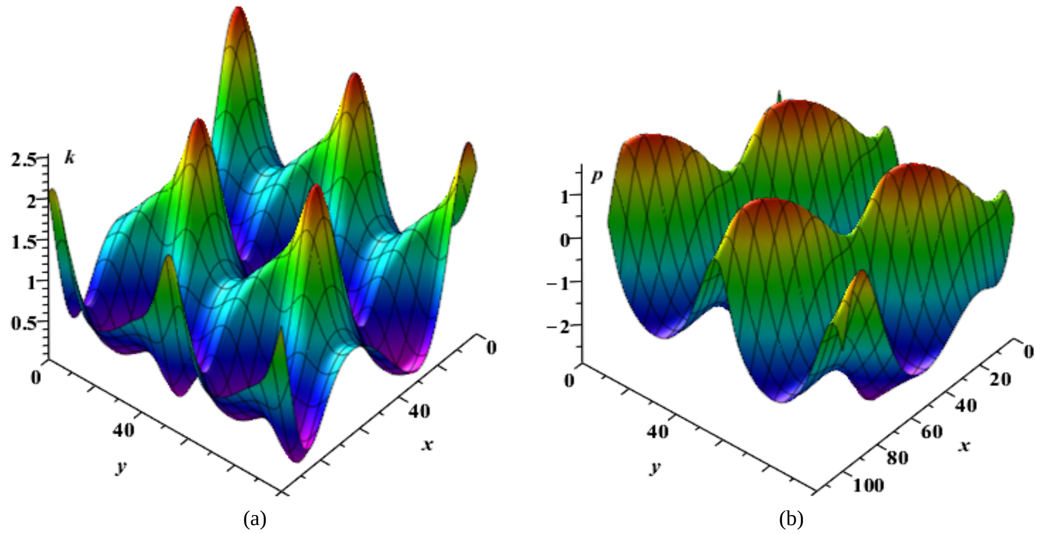

Figure 3. Kinetic energy (a) and dynamic pressure (b) of the lower cumulative flow.

tions. Invariance of the SKEF and SDEF structures with respect to differentiation is shown. The nonlinear algebra of the SKEF structures is developed. The non-orthogonal SKEF structural basis for harmonic functions is constructed and decompositions of the fluid-dynamic variables in this basis are obtained. The conservative system of N internal waves is neutrally stable with respect to M wave perturbations, and it is not affected by viscous dissipation.

The computational method of decomposition in invariant structures continues the analytical methods of separation of variables, undetermined coefficients, and series expansions [3] [5] [10]-[12]. In the current paper, the method of decomposition in invariant structures is extended into three dimensions. By this method, the vorticity and continuity PDEs are reduced to 12 homogeneous ODEs of first order and four linear AEs, the Helmholtz PDEs to 12 inhomogeneous ODEs of first order and 16 linear AEs, the Lamb-Helmholtz PDEs to four inhomogeneous ODEs of first order and eight linear AEs, and the Bernoulli equation is reduced to a linear equation in the SKEF and SDEF structures both for the upper and lower flows. To summarize, the system of four Navier-Stokes PDEs is reduced to the linear system of 57 equations, including 28 ODEs and 29 AEs.

Experimental discovery and theoretical proof of the exact solutions are implemented through experimental programming in Maple™ with lists of equations and expressions for numerical indices and numeric $N = 3$ by 33 developed procedures of 1748 code lines and theoretical programming with symbolic general terms, symbolic indices, and code-generated names of structural variables for symbolic N by 33 developed procedures of 1938 code lines. The developed procedures allow for theoretical derivations in the environment of novel COMP MATH data structures—the SKEF, SDEF, and SKEF-SDEF structures that are extended in this paper into three dimensions.

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