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The Periodic Table of Primes

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Abstract

Over millennia, nobody has been able to predict where prime numbers sprout or how they spread. This study establishes the Periodic Table of Primes (PTP) using four prime numbers 2, 3, 5, and 7. We identify 48 integers out of a period $2\times3\times5\times7=210$ to be the roots of all primes as well as composites without factors of 2, 3, 5, and 7. Each prime, twin primes, or composite without factors of 2, 3, 5, and 7 is an offspring of the 48 integers uniquely allocated on the PTP. Three major establishments made in the article are the Formula of Primes, the Periodic Table of Primes, and the Counting Functions of Primes and Twin Primes.

Keywords

Primes, Composites, the Periodic Table of Primes

1. Introduction

One of the greatest theorems of mathematics states that a composite integer can be represented uniquely as a product of primes. Today primes as roots of integers are studied and applied widely to data science, cryptography [1], systems reliability design [2], etc. For a long time, many said these integer roots grew like weeds among natural numbers, and nobody could predict where the next primes may sprout [3]. Many believe that primes are unpredictable. Oliver and Soundararajan [4] investigated the distribution of consecutive primes, while Luque and Lacasa [5] reported patterns of the last digit of primes. Wang [6] adopted pictures to search for regularities of some of the primes, but with significant bias.

Owing to the lack of insight, there is no effective computable formula for counting functions of primes and twin primes. Studies published to predict primality and pattern of primes are ad hoc with tremendous limitations and uncertainties.

Some [5] observed that primes located near each other tend to avoid repeating their last digits, which indicates that primes are not distributed as randomly as theorists often assume. Others such as Tóth [7] found the existence of the primes in k-tuple where the difference of two neighboring primes stays the same. However, few are able to explain what and where these tuple primes occur, nor can they predict the longest tuple primes within an interval.

Initiated by Gauss, there have been studies on counting the total number of primes and twin primes within an interval, but only approximations were obtained [8] [9]. Since primes are the roots of integers, can we identify the roots of primes? Can we find rules for generating composites; and if possible, may we say that, in some aspects, primes are predictable?

Li *et al.* [10] used 2, 3 and 5 in building a universal color system C_{235} to unify RGB (a light color frame) and CMYK (a pigment color frame). C_{235} represents colors R(red), G(green), and B(blue) by primes 2, 3, and 5, respectively. Consequently, C(cyan), M(magenta), Y(yellow), and K(key black) are represented by $3\times5=15$, $2\times5=10$, $2\times3=6$, and $2\times3\times5=30$, respectively. Through this transformation, all colors are representable by 7 root numbers of 2, 3, 5, 6, 10, 15, and 30. These root numbers encode millions of colors on a color wheel [10].

Inspired by C_{235}, we intend to:

- find a set of integers which serve as the root of primes and twin primes,
- establish the Formula of Primes and build the Periodic Table of Primes (PTP) that allocates primes, and
- form the Counting Functions of Primes and Twin Primes.

Also accomplished include predicting within an interval, the largest k-tuple primes [8] with the same difference.

1.1. Steps and Notation Adopted to Build the PTP

First, we define a concise strategy by selecting the roots of primes greater than 10 and composites without factors of 2, 3, 5, and 7, and then form the Cyclic Table of Composites (CTC) by identifying the locations of such composites, followed by advancing the Formula of Primes and building the PTP. The procedures are summarized in 4 steps:

1) Selecting the roots and cycles

Adopting the first four primes 2, 3, 5, and 7, we take $2\times3\times5\times7=210$ as the length of a period. Within the interval [11, 211], we sort out 48 integers of primes or composites that do not contain factors of 2, 3, 5, and 7. These 48 integers, to generate primes and composites, are considered as the roots denoted by $r_1=11$, $r_2=13$, $r_3=17$, $\cdots$, $r_{23}=103$, $r_{24}=107$, $\cdots$, $r_{48}=211$. These r_is are placed on the left column of a table.

2) Developing the Cyclic Table of Composites (CTC)

The CTC consists of multiple 48 × 48 tables $\left[l(i,j)\right]$ derived from the positions of composites without factors of 2, 3, 5, and 7.

Within an interval of $\left[48(\theta-1)+1, 48\theta\right]$, where $\theta \in N_+$ is considered a cycle, there are 48 integers which do not contain factors of 2, 3, 5, and 7. These $48\times\theta$ integers are either primes or composites, denoted as $q_1 = r_1 = 11$, $q_2 = r_2 = 13$, $q_3 = r_3 = 17$, $\cdots$, $q_{48} = r_{48} = 211$, for cycle 1; $q_{49} = q_1 + 210 = 221$, $q_{50} = q_2 + 210 = 223$, $\cdots$, $q_{96} = q_{48} + 210 = 421$, for cycle 2; $\cdots$;
$q_{48(\theta-1)+1} = q_{48(\theta-2)+1} + 210 = q_1 + 210(\theta-1)$,
$q_{48(\theta-1)+2} = q_{48(\theta-2)+2} + 210 = q_2 + 210(\theta-1)$, $\cdots$, and
$q_{48\theta} = q_{48(\theta-1)} + 210 = q_{48} + 210(\theta-1)$, for cycle θ. We call q_1 to q_{48} segments of cycle 1, q_{49} to q_{96} segments of cycle 2, $\cdots$, and $q_{48\times(\theta-1)+1}$ to $q_{48\theta}$ segments of cycle θ. The q_js are placed on the top row of a CTC table of θ cycles.

Formulating the CTC of the first cycle, namely, CTC(1), we observe the dual effect existing in $\left[l(i,j)\right]$, where index i refers to r_i and j refers to q_j, such that in each row of this table, pairs of entries share the same value of elements $l(i,j)$s. We also find for each row in the CTC(1), there is another row complementary to it, calling it the mirror effect. Specifically, there exist the mirror effects between the 23 top rows and the next 23 rows as well as between the 47th row and the 48th row of CTC(1). We define the basic CTC (CTC(basic)) to be a 24 × 48 table of $\left[l(i,j)\right]$, $i = 1, 2, \cdots, 23, 47$ and $j = 1, 2, \cdots, 48$.

Utilizing the mirror effect, we find the complementary CTC (CTC (complementary)) a 24 × 48 table made of $\left[l(i,j)\right]$ for $i = 24, 25, \cdots, 46, 48$ and $j = 1, 2, \cdots, 48$. Combining the CTC (basic) and the CTC(complementary), we form CTC(1).

We observe further an intermedia effect in CTC(1) between different cycles of composite tables where each is 48 × 48 in size.

3) Advancing the Formula of Primes

By deleting all composites located by CTC in an interval, we develop the Formula of Primes.

Given a positive integer b with b – 211 being a multiplier of 210, an integer $\alpha \in [0, b]$ without factors of 2, 3, 5, and 7 is a prime if and only if there exists an $i \in \{1, 2, \cdots, 48\}$ such that $\alpha = r_i + 210k$, for some integer k with $k + 1 \notin L_b(i)$, where $L_b(i)$ is a set of $l(i,j)$s which satisfy some conditions of b and θ described in the Results section.

4) Building the Periodic Table of Primes (PTP)

Denote PTP(k^0, k') as a periodic table of primes from period k^0 to k', $k^0 < k' \in N_+$. PTP(k^0, k') is a $48\times(k' - k^0 + 1)$ table composed of all primes within an interval $[a, b]$, for $a = 11 + 210\times k^0$ and $b = 211 + 210\times k'$. Denote PTP$^+$(k^0, k') as a table composed of primes in PTP(k^0, k') plus composites without factors 2, 3, 5, and 7 within the interval $[a, b]$.

The steps of building a PTP are depicted in Figure 1. Starting from the CTC (initial), we form a CTC (initial dual). By combining the CTC (initial) and the CTC (initial dual), we obtain the CTC (basic) which is outlined in the Supplement under Establishing the CTC (basic) by the CTC (initial). Once CTC (1) is deduced from CTC (basic), all subsequent CTC (2), CTC (3), …, CTC (θ) are

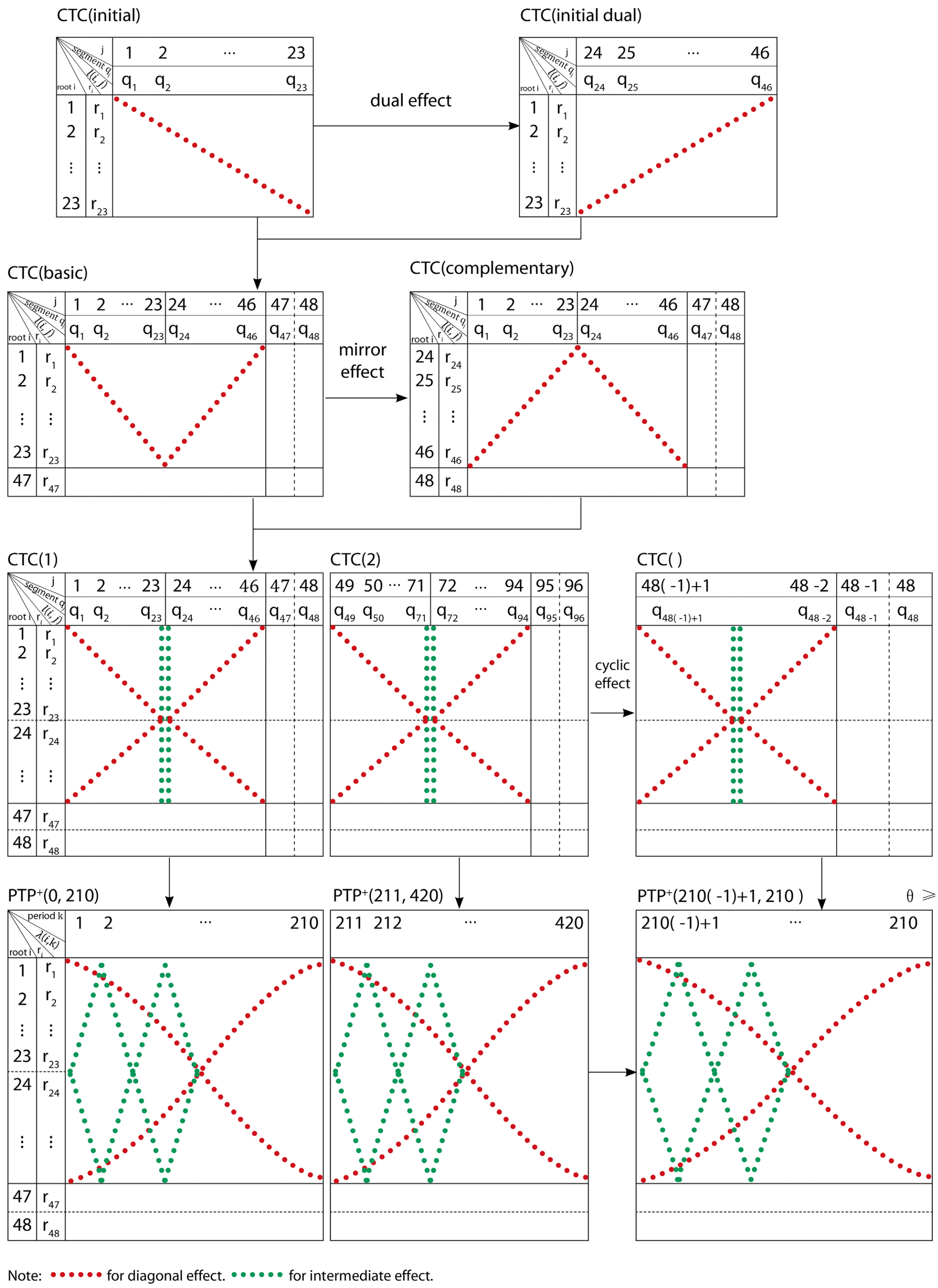

Figure 1. Schematic process of forming the PTP.

readily available. Utilizing CTCs, we generate PTP$^+$(k^0, k'), where we delete all composites to obtain PTP(k^0, k'). The red and green points in **Figure 1** represent the diagonal and intermediate effects, respectively, and also show the cyclical pattern on CTC and the periodical pattern on PTP$^+$.

1.2. Note

Lei *et al.* [11] held that the evolution of Chinese historical dynasties can be mapped by the properties of elements' electrons in the Periodic Table of Chemical Elements. Wang [12] claimed that irregular primes are the survivors of natural numbers after regular decimations by primes.

There are various sieve methods to locate primes. In coming up with a solution for predicting primes, many have searched for a particular pattern of primes distributions. For example, Holt [13] mentioned structure to cycles due to recursion and created exact relative population models for all gaps up to a certain level. Dastych found a "mirroring" effect of numbers such as 2 × 3, 2 × 3 × 5 and 2 × 3 × 5 × 7 being represented by a wheel which were already in existence [14]. His algorithm originated from playing with Goldbach's conjecture, but became useless when the numbers grew large. Others may have expressed their wishes to organize primes listing over the years. Although we tried to be thorough and are blessed by many professional colleagues in reaching this unique PTP, we could miss non-disclosed endeavors made by individuals beyond the publicly accessible domain.

2. Transformation and Observations

Without loss of generality, discussions in this section on the CTC, the CTC(basic) and the CTC(complementary) are referred to cycle 1, namely, CTC(1), unless stated otherwise.

In CTC, consisting of 48 rows and 48 columns, denote r_i as the i^{th} row and q_j the j^{th} column.

Let

$$S_1 = \{r_1, r_2, \cdots, r_{23}, r_{47}\} = \{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 209\}$$

$$S_2 = \{r_{24}, r_{25}, \cdots, r_{46}, r_{48}\} = \{107, 109, 113, 121, 127, 131, 137, 139, 143, 149, 151, 157, 163, 167, 169, 173, 179, 181, 187, 191, 193, 197, 199, 211\}$$

and

$$S = S_1 \cup S_2 = \{r_1, r_2, \cdots, r_{48}\} = \{q_1, q_2, \cdots, q_{48}\}.$$

2.1. Transformation

Note that r_is and q_js are integers in $[11, 211]$ without factors of 2, 3, 5, and 7. Let (i, j) be at a position in the CTC and $l(i, j)$ the corresponding entry of point (i, j), where $1 \le i, j \le 48$, determined by

$$l(i, j) = 1 + \left(q_j \times q_{\hat{j}} - r_i\right) / 210. \quad (1)$$

Given the i^{th} row and j^{th} column, we observe a unique $q_{\hat{j}} \in S$ that couples with q_j such that $q_j \times q_{\hat{j}} - r_i$ is a multiplier of 210. Both q_j and $q_{\hat{j}}$ depend on r_i or the i^{th} row.

2.2. Three Statements

An outline of CTC is shown in **Figure 2**. Various effects in CTC are described by the following statements. Presented in **Figure S1(a)** along with an illustrative example are given in the Supplement under Example for **Figure S1(a)**. Observing **Figure S1(a)**, note that $l(i,i) = q_i + 1$, $i = 1,2,\cdots,23,47$. We identify the diagonal effect in CTC in Statement 1 below.

Statement 1 (The Diagonal Effect of the CTC)

In the CTC, $l(j,j) = q_j + 1$**, for** $j = 1,2,\cdots,48$. **Moreover,** $l(j,j) \ge l(i,j)$ **for** $i,j = 1,2,\cdots,48$ **and** $i \ne j$.

Analyzing the distribution of $l(i,j)$ in the i^{th} row of **Figure S1(a)**, we find the dual effect between $l(i,j)$ and $l(i,\hat{j})$, described in Statement 2 below.

Statement 2 (The Dual Effect of the CTC)

In the CTC, for any $l(i,j)$, $i,j = 1,2,\cdots,48$ **and** $i \ne j$**, there exists a**

$\hat{j} \in \{1,2,\cdots,48\}$

such that $l(i,j) = l(i,\hat{j})$.

Statement 2 implies that given $l(i,j)$, we know $l(i,\hat{j})$, under which situation j and $\hat{j}$ are dual to each other. Take $i = 1$ and $r_1 = 11$ for instance, $l(1,5) = l(1,8) = 5$, where $q_5 \times q_8 = 23 \times 37 = 851 = 11 + 210(5-1)$. Notice that when $\hat{j} = j$, q_j^2 is a descendant of r_i and $l(i,j)$ appears in the i^{th} row only once. For instance, $l(48,9) = 8$ appears only once and

$41^2 = 1681 = 211 + 210 \times (8-1)$.

Further elaboration of the dual effect and associated examples are given in the Supplement under Elaboration of the Dual Effect of the CTC (basic). Analyzing **Figure S1(a)** and **Figure S1(b)**, we also identify the mirror effect between each row of the CTC(basic) and its complementary row in CTC deduced below.

Statement 3 (The Mirror Effect of the CTC)

The mirror effect in the CTC exists between pairs of all rows such that

(i) $$l(47-i,j) + l(i,j) = \begin{cases} q_j + 1, \text{ if } i \ne j \text{ and } i + j \ne 47 \\ 2q_j + 1, \text{ if } i = j \text{ or } i + j = 47 \end{cases},$$

for $i = 1,2,\cdots,23$, $j = 1,\cdots,48$

(ii) $$l(47,j) + l(48,j) = \begin{cases} q_j, \text{ if } j = 1,\cdots,46 \\ 2q_j, \text{ if } j = 47,48 \end{cases}$$

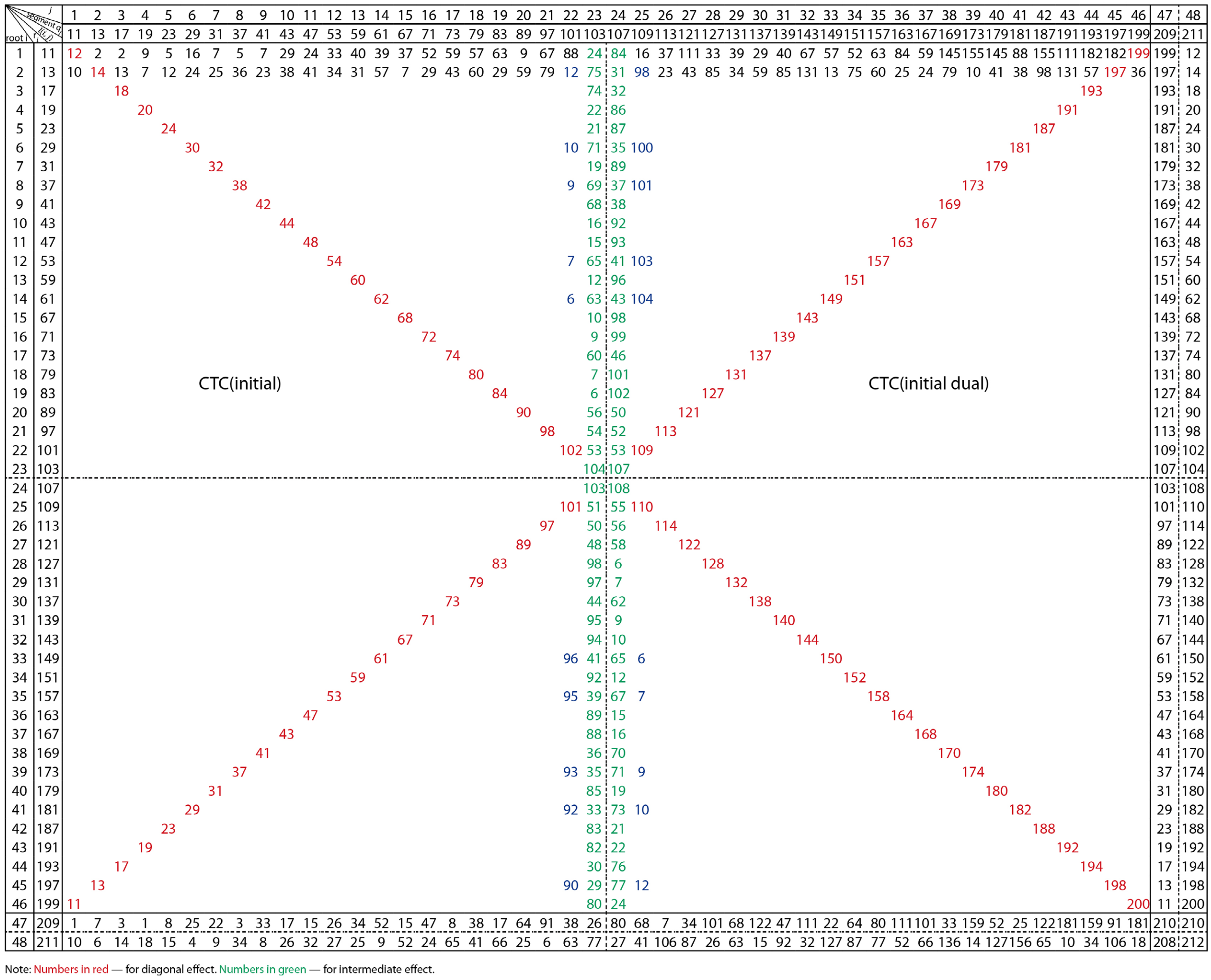

Figure 2. An outline of CTC(1).

Statement 3 implies that if we know $l(i,j)$ for $1 \le i \le 23$, then we can find $l(47-i,j)$. Similarly, if we know $l(47,j)$, then we find $l(48,j)$. Following Statement 3, we form the CTC (complementary), the complementary of the CTC (basic).

CTC (complementary) is presented as **Figure S1(b)**, in the Supplement. Merging the CTC (basic) and the CTC (complementary), we obtain the CTC (1) as the CTC for cycle 1, referring to **Figure 2**.

Consider CTC in **Figure 2**. There are two red diagonals shown in **Figure 2**, which further elaborates **Figure S1(a)** and **Figure S1(b)**, The first diagonal is $l(1,1)=12$, $l(2,2)=14$, $\cdots$, and $l(46,46)=200$, and the second one is $l(46,1)=11$, $l(45,2)=13$, $\cdots$, and $l(1,46)=199$. The two cross lines in **Figure 2** are obvious due to the mirror effect. $l(47,j)$ and $l(48,j)$, $j=1,2,3,\cdots,48$, are mirrors with each other. An example is given in the Supplement under An Associated Example for Statement 3.

Figure S1(a) and **Figure S1(b)** show another distribution of $l(i,j)$ on CTC, named the intermedia effect, which happens at some symmetrical columns, described in the Supplement under Statement and Proof on the Intermedia effect in the CTC. Both the diagonal effect and the intermedia effect on the CTC are helpful in showing how to transfer CTC to PTP. In **Figure 2**, the red and green numbers represent the diagonal and the intermediate effects, respectively.

Proofs for the above Statements are given in the Supplement under Proofs of the Statements.

3. Results

This study provides three major results: the Formula of Primes, the Periodic Table of Primes, and the Counting Functions of Primes and Twin Primes.

Given a positive integer $b = 211 + 210 \times \theta^* \hat{\theta}^*$, $\theta^*, \hat{\theta}^* \in N_+$, for $i \in \{1, 2, \cdots, 48\}$, define

$$L_b(i) \equiv \Big\{ l(i,j) + q_j(\hat{\theta}-1) + q_{\hat{j}}(\theta-1) + 210(\theta-1)(\hat{\theta}-1) \le \theta^* \hat{\theta}^* + 1 \;|\; j = 1, 2, \cdots, 48, \text{ with the corresponding } \hat{j} \text{ defined in Expression } (1), \theta = 1, 2, \cdots, \theta^* \text{ and } \hat{\theta} = 1, 2, \cdots, \hat{\theta}^* \Big\} \quad (2)$$

For the case $\hat{\theta}^* = 1$, then

$$L_b(i) \equiv \left\{ l(i, j + 48(\theta-1)) \le \theta^* + 1 \;|\; j = 1, 2, \cdots, 48, \text{ and } \theta = 1, 2, \cdots, \theta^* \right\} \quad (2')$$

Notice that $L_b(i)$ is a set consisting of $l(i,j) \le \Delta_{i,\theta+1}$, for $j = 1, 2, \cdots, 48\theta$, where $\Delta_{i,\theta} = \min_j \{ l(i,j) \,|\, j = 48(\theta-1)+1, 48(\theta-1)+2, \cdots, 48\theta \}$ for $\theta \in N_+$, and $210\Delta_{i,\theta+1} \ge b$.

Also $l(i, j+48(\theta-1)) = l(i,j) + (\theta-1) q_{\hat{j}}$, where $q_{\hat{j}}$ is derived from Expression (1). For any positive integer α containing no factors of 2, 3, 5, and 7, α must have a unique root $r_i \in S$ such that $\alpha - r_i$ is a multiple of 210. Moreover, if α is not a composite number, then it is a prime.

An example for $L_b(1)$ where $\hat{\theta} = 1$ can be found in Supplement for the Text "7. Justification for Establishing $L_b(1)$ and **Figure S2**".

3.1. The Formula of Primes

We summarize the Formula of Primes below:

The Formula of Primes

An integer $\alpha \in [11, b]$ containing no factors of 2, 3, 5, and 7 is a prime if and only if there exists $r_i \in S$ and $k \in N_0$ such that $\alpha = r_i + 210k$, with $k + 1 \notin L_b(i)$, $i = 1, 2, \cdots, 48$.

A list of $L_b(1)$ for various b up to 44521 and $\hat{\theta} = 1$ is given in **Figure S2**, which is presented in the Supplement.

3.2. The Periodic Table of Primes

According to CTC(θ) and the Formula of Primes, we specify the Periodic Table of Primes below.

The Periodic Table of Primes from period k^0 to period k'
For given $k^0 \in N_0$, $k' \in N_+$, and $k' > k^0$, PTP(k^0, k') is a $48 \times (k' - k^0 + 1)$ table $[\lambda(i, k+210)]$, where $\lambda(\cdot)$ is an integer such that

$$11 + 211 \times k^0 \le \lambda(i,k) = r_i + 210k \le 211(k'+1),$$

for $k \in N_0$ with $k+1 \notin L_b(i)$ and $b = 211(k'+1)$.

Denote PTP⁺(k^0, k') as the table composed of primes in PTP(k^0, k') plus composites without factors of 2, 3, 5 and 7, within an interval $[11 + 211 \times k^0, 211(k'+1)]$.

Figure 3 is an outline of PTP⁺(0, 210), which includes PTP(0, 210) and composites without factors of 2, 3, 5, and 7 within the interval $[11, 1 + 210^2]$ converted from Figure 2. Figure 3 has 48 rows and 211 columns, for $i = 1, 2, \cdots, 48$, and $k = 0, 1, 2, 3, \cdots, 210$. $\lambda(i,k)$ is an integer of the entry (i,k), computed from $l(i,j)$ of Figure 2. The two dotted red curves in Figure 3 are configured from two diagonal red lines of Figure 2, *i.e.*, 12–14–18···198–200 and 11–13–17···197–199. The two dotted green diamond-shape lines in Figure 3 come from two green lines of Figure 2 of 24–75–74···29–80 and 84–31–32···77–24.

Figure 4 is a realization of PTP⁺(0, 10) including primes and composites without factors of 2, 3, 5, and 7. Further elaboration is seen in the Supplement under Numerical Illustrations for The Periodic Table of Primes. According to Gauss [8] [9], the number of primes no more than *b* is approximately $b/\ln b$. We find further the exact Counting Function of primes $\pi(b)$ when *b* – 211 is a multiplier of 210.

3.3. The Counting Functions of Primes and Twin Primes

We use $\pi(b)$ and $\pi^*(b)$ to represent the Counting Function of Primes and the Counting Function of Twin Primes in $[1, b]$ for a natural number *b*, respectively.

The Counting Function of Primes
Given a positive integer $b = 211 + 210 \times k$,

$$\pi(b) = 4 + 48 \times (k+1) - \sum_{i=1}^{48} |L_b(i)|. \quad (3)$$

Currently, few predict the number of twin-prime pairs in an interval [2, *b*]. For *b* – 211 as a multiplier of 210, denote $T(i, i+1)$ as the set of twin-prime pairs on PTP($0, \lceil b/211 \rceil$), *i.e.*, a set of *k* with $k+1 \notin L_b(i)$ and $k+1 \notin L_b(i+1)$, where *i* and *i*+1 are for the *i*th and the (*i*+1)st row on the PTP, respectively. Our study finds the exact $\pi^*(b)$ below:

The Counting Function of Twin Primes

$$\pi^*(b) = 3 + \sum_{i \in W} |T(i, i+1)|, \quad (4)$$

where $W = \{1,3,6,9,13,16,22,24,30,33,37,40,43,45,47\}$.

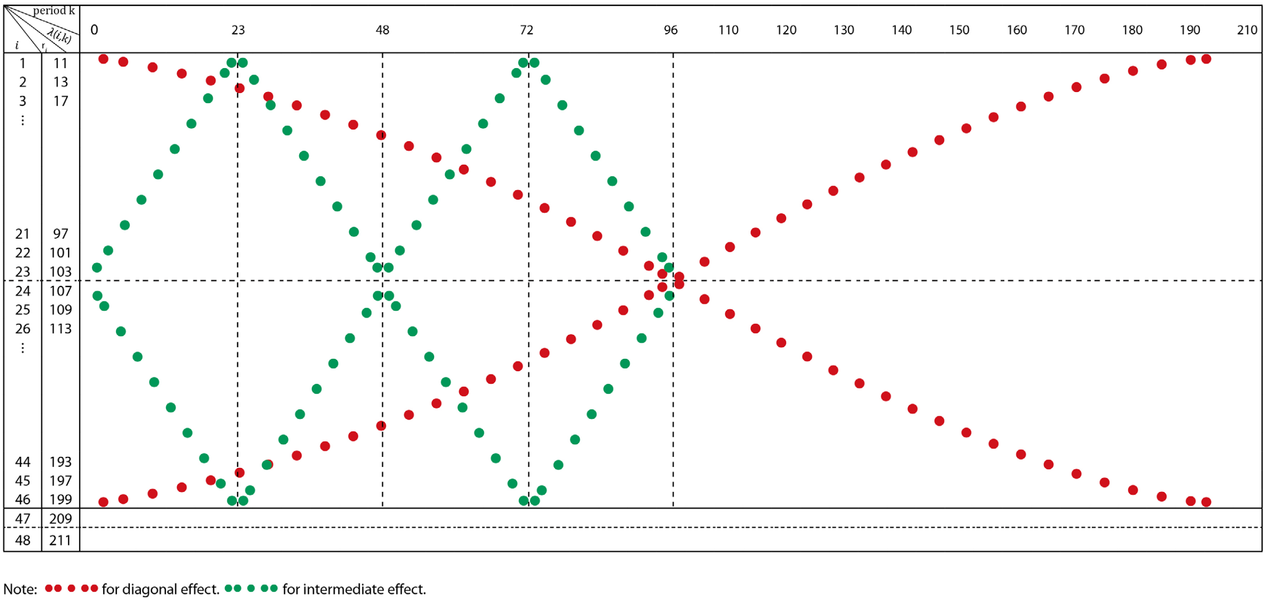

Figure 3. An outline of PTP⁺(0,210).

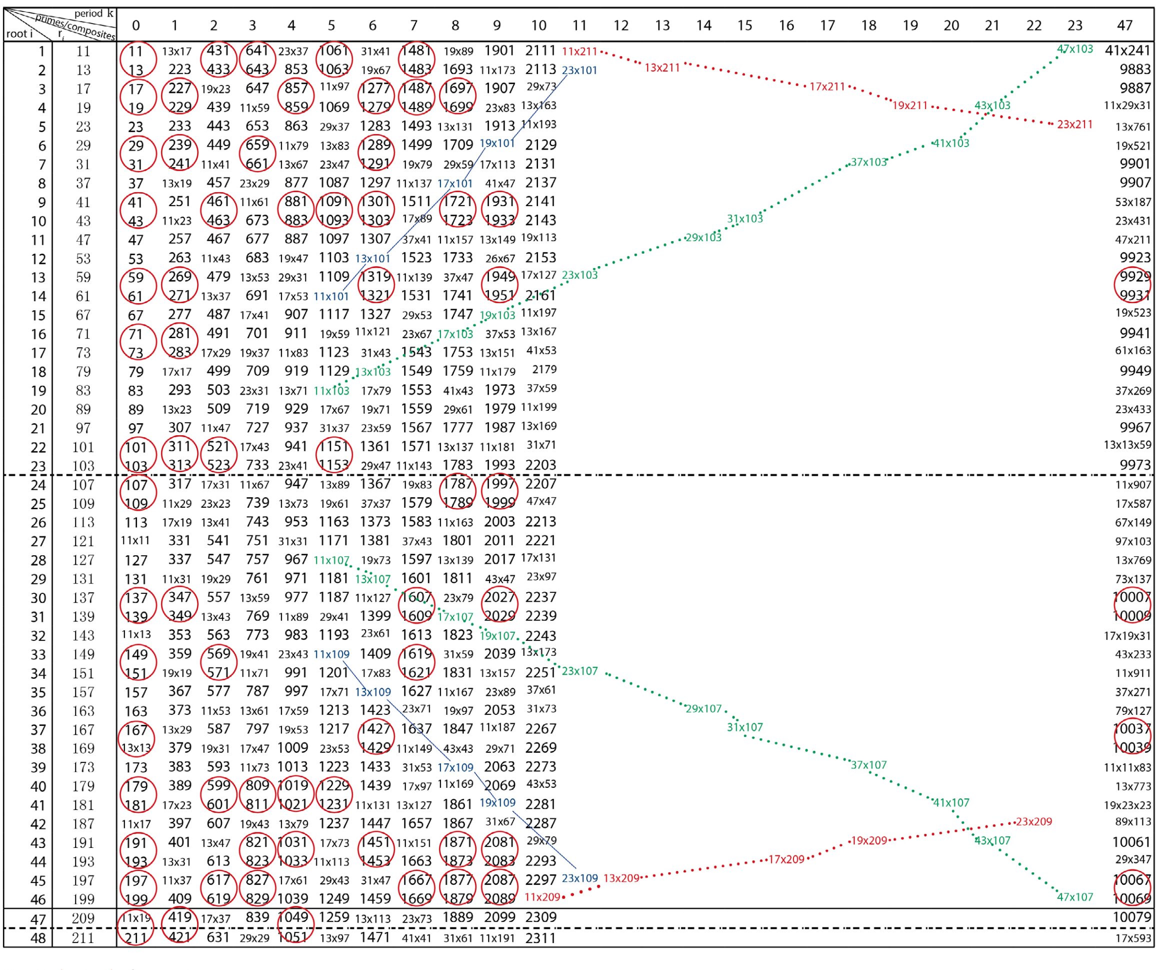

Figure 4. A partial table of PTP⁺(0,48).

Elaborations of Expressions (3) and (4) are given in the Supplement under Elaborations of Expressions (3) and (4), and examples of computing $\pi(b)$ and $\pi^*(b)$ are illustrated in the Supplement under Examples for the Predictions of Primes by the PTP.

4. Discussions

The unpredictability of prime numbers forms the basis of many applications, one being encryption called the RSA algorithm [1]. However, neither the ancient Sieve of Eratosthenes nor the modern Sieve of Atkin's algorithms [15] have ever elaborated the physical meaning of finding primes. Torquato, Zhang and Courcy-Ireland [16] claimed that they found a physical structure pattern hidden in the distribution of prime numbers, but that discovery still didn't explain the essence of prime numbers.

This paper identifies 48 natural numbers between 11 and 211, which do not contain factors of 2, 3, 5, and 7, to be the roots for generating all primes and composites without factors of 2, 3, 5, and 7. The locations of such composites exhibit periodic and cyclic properties, as represented by the CTC, which enable us to eliminate them for finding primes, as represented by the PTP. Treating the 48 roots as the genes of prime numbers, we can easily find the next prime of any given prime number and identify the next pair of twin primes. Our findings provide a platform to study many primes-related problems. No primes, twin primes or primes-related issues can ever surface if such issues are not rooted to the 48 integers. After all, prime numbers are not as random as many believe.

We form the CTC, followed by the PTP. All these present the primes effectively and with physical meaning. In addition, we can count the exact numbers of primes and twin primes within an interval. Discussed below are some further thoughts.

1) Instead of choosing 2, 3, 5 and 7, one may add 11, 13, or more primes to generate the roots. By so doing, the PTP will gradually become gigantic, too complicated, and too difficult to visualize, although likely more effective. If one is interested in the behaviors of super-large primes, one could find it useful in such large tables, which may be extended to infinite. On the other hand, one may choose 2, 3, and 5 to generate 8 roots for a small prime table.

2) The PTP is helpful in understanding many unclear phenomena. For instance, it explains a troublesome observation [5] that for a given prime with the last digit of 1, the chance of its next prime to have 1 as its last digit is much less than that of 3 or 7 or 9. From the Formula of Primes, if a given prime is 221 (*i.e.*, $\theta = 2$, $r_i = 11$), then the most possible near primes should be firstly 223 (*i.e.*, $\theta = 2$, $r_i = 13$), followed by 227, 229, 233, 239, and 241.

3) From the 48 roots identified, we find no triplet or higher multiples of primes existing in the roots. Therefore, there will be no triplet primes found in future generations. Likewise, all twin primes will appear exactly at the parallel locations as those appearing in the 48 roots of the PTP. This implies that all twin-prime pairs are descendants of 15 pairs of twin primes or composites. In

fact, an equal chance is found in the last digit of the 48 roots for 1, 3, 7, and 9. Therefore, the last digits for all primes will each have a 25% chance of being 1, 3, 7, and 9 when primes go to infinity.

4) This is the first time that a visualizable prime table is built with proofs using a manageable set of primes as the basis for making useable influences and clarifying some questions of interest in existence for years. Just like reported, openly or not, by Oliver and Soundaranjan [4], Wang [12], Holt [13], Dastych [14], etc., no conclusion has been drawn on establishing a compact table for primes of any kind.

In contrast, from building the CTC, we demonstrate that every prime has an ancestor among 48 integers which include 43 primes and 5 composites, and every twin-prime pair comes from an ancestor of 15 pairs of these 48 integers. We develop the algorithms and give illustrating examples in the Supplement using the PTP from which we draw several inferences and present some useful applications.

5) Beyond the fundamental investigation, this study is due to part of our efforts in exploring various applications of primes, including systems reliability design [2], and building a color system C_{235} to unify RGB and CMYK and to encode millions of colors on a color wheel [10]. The universal color system wins the Special Prize and the Gold Medal with Congratulations of the Jury at the 49th International Exhibition of Inventions of Geneva.

5. Conclusions

This paper selected 48 integers as the roots to generate primes and composites without the factors of 2, 3, 5, and 7. We constructed a composite table, CTC, further observing the diagonal effect, the dual effect and the mirror effect. Based on the CTC, this paper introduced the first closed form expressions for the Formula of Primes and the Periodic Table of Primes. The Counting Functions of Primes and Twin Primes are then readily deduced. Related mathematical proofs and computations are exercised to testify the correctness of the above statements.

While there exist proven and unproven concepts, approaches and analysis in the literature, our study clearly shows that any prime except 2, 3, 5, and 7 can be uniquely rooted to one of 48 natural numbers between 11 and 220 in cycles of length 210. Moreover, any twin-prime pair can be uniquely rooted to a pair of the same 48 roots in cycles of length 210. We show that all composite numbers with no factors of 2, 3, 5, and 7 appear in cyclic manner. To the best of our knowledge, these fundamental findings are the first to systematically and concretely address in the open literature. People may have used terminologies similar to this study, but neither similar results nor close form solutions have ever been presented.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Supplement for the Text

1. Establishing the CTC (basic) by the CTC (initial)

Specifically, we adopt the following steps to establish the CTC (basic):

a) From the CTC (basic) of Figure 1, we find 23 × 23 CTC (initial) table for the first cycle. In the CTC (initial), its entries of $\left[l(i,j)\right]$, for $i=1,2,\cdots,23$ and $j=1,2,\cdots,23$, are both starting from 11; namely, the left column contains $r_1,r_2,\cdots,r_{23}$ and the top row contains $q_1,q_2,\cdots,q_{23}$.

b) Applying the dual effect, we form another 23 × 23 CTC (initial dual) table from CTC (initial). In this new table, its entries of $\left[l(i,j)\right]$, for $i=1,2,\cdots,23$ and $j=24,25,\cdots,46$, are the first 23 roots starting from 11 for "*i*"s and the 24th to 46th roots for "*j*"s, respectively. Namely, the left column contains $r_1,r_2,\cdots,r_{23}$, and the top row contains $q_{24},q_{25},\cdots,q_{46}$.

c) Add r_{47} as the end of the 24th row and add q_{47} and q_{48} as the end of the 47th and the 48th columns. Namely, the left column contains r_{47} as the last index and the top row contains q_{47}, q_{48} as the last two indices.

d) Combine tables developed by a, b, and c steps to complete CTC(basic).

As shown in Figure 1, there are the CTC(initial), the CTC(initial-dual), the CTC(basic), the CTC(complementary), and CTC(1).

$$\text{CTC}(\text{initial})=\left\{l(i,j)\,|\,i=1,2,\cdots,23;\,j=1,2,\cdots,23\right\},$$

$$\text{CTC}(\text{initial-dual})=\left\{l(i,j)\,|\,i=1,2,\cdots,23;\,j=24,\cdots,46\right\},$$

$$\text{CTC}(\text{basic})=\left\{l(i,j)\,|\,i=1,2,\cdots,23,47;\,j=1,2,\cdots,48\right\},$$

$$\text{CTC}(\text{complementary})=\left\{l(i,j)\,|\,i=24,25,\cdots,46,48;\,j=1,2,\cdots,48\right\},$$

$$\text{CTC}(1)=\left\{l(i,j)\,|\,i,j=1,2,\cdots,48\right\}.$$

2. Example for Figure S1(a)

Given $q_1=11$, compute $l(1,1)$ as follows:

Since $l(1,1)=1+\frac{q_1\times q_{\hat{1}}-r_1}{210}$, and the unique $q_{\hat{1}}$ to let $\frac{11\times q_{\hat{1}}-11}{210}$ be an integer is

$$q_{\hat{1}}=211,\ l(1,1)=1+\frac{11\times 211-11}{210}=1+11=12.$$

Similarly, we find

$$l(2,1),l(3,1),l(4,1),l(5,1),l(6,1),\cdots,l(23,1),l(47,1)=(10,6,4,11,5,\cdots,8,1).$$

3. Elaboration of the Dual Effect of the CTC (basic)

In the CTC(1), the dual effect made in Statement 2 can be further elaborated below between $l(i,j)$ and $l(i,\hat{j})$.

(i) For $l(i,j)=l(i,\hat{j})$, if $1\le j,\hat{j}\le 23$, then

$$l(i,47-j)=l(i,47-\hat{j})=1+\frac{q_{47-j}\times q_{47-\hat{j}}}{q_j\times q_{\hat{j}}}\left(\frac{r_i}{210}+l(i,j)-1\right)-\frac{r_i}{210}.$$

(ii) For $l(i,j)=l(i,\hat{j})$**, for** $1\le j\le 23$ **but not for** $\hat{j}$**, then** $24\le \hat{j}\le 46$ **and** $q_{\hat{j}}=\frac{r_i+210[l(i,j)-1]}{q_j}$.

By utilizing Statement 2, we can generate a new table CTC(initial dual) in the Introduction from CTC(initial). Taking the first row, *i.e.* $i=1$ and $r_1=11$ as an example (**Figure 2**):

(i) Since $l(1,2)=l(1,3)=2$, we have

$$l(1,47-2)=l(1,47-3)=1+\frac{q_{45}\times q_{44}}{q_2\times q_3}\left(\frac{11}{210}+l(1,2)-1\right)-\frac{11}{210}=182$$

Similarly, since $l(1,5)=l(1,8)=5$, we have

$$l(1,47-5)=l(1,47-8)=1+\frac{q_{42}\times q_{39}}{q_5\times q_8}\left(\frac{11}{210}+l(1,5)-1\right)-\frac{11}{210}=155$$

(ii) For $i=1$ and $j=1,6,10,12,\cdots,19,21,22$, we do not have $1\le \hat{j}\le 23$ to fit $l(1,j)=l(1,\hat{j})$, then we compute $l(1,\hat{j})$ for $24\le \hat{j}\le 46$.

When $j=1$, $q_1=11$ and $l(1,1)=12$,

since $\frac{r_1+210(l(1,1)-1)}{q_1}=\frac{11+210(12-1)}{11}=211=q_{48}$, we have

$$\hat{1}=48 \text{ and } l(1,48)=l(1,1)=12$$

When $j=6$, $q_6=29$ and $l(1,6)=16$,

since $\frac{r_1+210(l(1,6)-1)}{q_6}=109=q_{25}$, we have

$$\hat{6}=25 \text{ and } l(1,25)=l(1,6)=16.$$

Similarly, we can predict $l(1,26)=l(1,15)=37$, $l(1,28)=l(1,12)=33$, $l(1,29)=l(1,14)=39$, $l(1,30)=l(1,10)=29$, $\cdots$, $l(1,41)=l(1,22)=88$.

4. An Associated Example for Statement 3

Comparing the CTC (basic) with the CTC (complementary) and referring to **Figure S1(a)** and **Figure S1(b)**, it is clear that:

Given $j=1$ and $q_1=11$, we have $l(1,1)+l(46,1)=12+11=2q_1+1$.

Given $j=2$ and $q_2=13$, we have $l(2,2)+l(45,2)=14+13=2q_2+1$.

Given $j=1$ and $q_1=11$, for $i=2$, we have $l(45,1)+l(2,1)=2+10=12=q_1+1$.

Given $j=12$ and $q_{12}=53$, for $i=1$, $l(46,12)+l(1,12)=21+33=54=q_{12}+1$; for $i=23$, $l(24,12)+l(23,12)=28+26=54=q_{12}+1$; for $i=47$, $l(47,12)+l(48,12)=26+27=53=q_{12}$.

5. Statement and Proof of the Intermedia Effect in the CTC(1)

Figure S1(a) and **Figure S1(b)** show another distribution of $l(i,j)$ on the CTC, named the intermedia effect, which happens at some symmetrical columns, described below.

The intermedia effect in the CTC(1)
In CTC(1), there exists the intermedia effect between columns 23 and 24 such that, for $i=1,2,\cdots,48$**,**
(i) if $l(i,23)+l(i,24)=108$**, then** $l(i,23)=l(47-i,24)$,
(ii) if $l(i,23)+l(i,24)=106$**, then** $l(i,23)+2=l(47-i,24)$.

For instances, in the case of green columns 23 and 24 of **Figure 2**, $l(19,23)=l(28,24)=6$, $\cdots$, $l(2,23)+2=l(45,24)=77$. Other column pairs also exhibit the intermedia effect. For example, in the case of columns 22 and 25, if $l(i,22)+l(i,25)=110$, then $l(i,22)=l(47-i,25)$.

Proof of the intermedia effect in the CTC

For the case of columns 23 and 24,

(i) From the mirror effect, we have $l(i,24)+l(47-i,24)=q_{24}+1=108$.

If $l(i,23)+l(i,24)=108$, then $l(i,23)=108-l(i,24)=l(47-i,24)$.

(ii) If $l(i,23)+l(i,24)=106$, then $l(i,23)+2=108-l(i,24)=l(47-i,24)$.

For the case of columns 22 and 25, from the mirror effect, we know $l(i,25)+l(47-i,25)=q_{25}+1=109+1=110$

If $l(i,22)+l(i,25)=110$, then $l(i,22)=l(47-i,25)$.

6. Proofs of the Statements

Proof of Statement 1

For each $i,j=1,2,\cdots,48$, notice that in CTC(1),

$$
\begin{aligned}
l(j,j) &= 1+\frac{q_j\times q_{\hat{j}}-r_j}{210}\\
&= 1+\frac{q_j(211-1)}{210}\\
&= 1+q_j,
\end{aligned}
$$

as $r_j=q_j$ and we choose $q_{\hat{j}}=q_{48}=211$.

For a given *j*, $r_i=q_j\times q_{\hat{j}}-\left(l(i,j)-1\right)\times 210$, for some $\hat{j}$. This means that a fixed r_i is the remainder of $q_j\times q_{\hat{j}}$ subtracting the maximum multiple of 210 smaller than $q_j\times q_{\hat{j}}$. Since $q_{\hat{j}}=q_{48}=211$ is the largest $q_{\hat{j}}$ hence $l(j,j)\geq l(i,j)$.

Proof of Statement 2

By definition, $l(i,j)=1+\frac{q_jq_{\hat{j}}-r_i}{210}$, $l(i,\hat{j})=1+\frac{q_{\hat{j}}q_j-r_i}{210}$.

Therefore, we have $l(i,j)=l(i,\hat{j})=1+\frac{q_j\times q_{\hat{j}}-r_i}{210}$

Proof of Statement 3

(i): In this proof, we use $q_{\hat{j}|i}$ to differentiate $q_{\hat{j}}$ given different r_i.

Notice that $r_{47-i} + r_i = 210$, $\forall i = 1, \cdots, 23$,

$l(47-i, j) = 1 + \frac{q_{j|i} q_{\hat{j}|(47-i)} - r_{47-i}}{210}$ for some $q_{\hat{j}|47-i}$ and

$l(i, j) = 1 + \frac{q_{j|i} q_{\hat{j}|i} - r_i}{210}$ for some $q_{\hat{j}|i}$.

Hence,

$$l(47-i, j) + l(i, j) = 2 + \frac{q_{j|i}\left(q_{\hat{j}|(47-i)} + q_{\hat{j}|i}\right) - 210}{210}$$
$$= 1 + \frac{q_{j|i}\left(q_{\hat{j}|(47-i)} + q_{\hat{j}|i}\right)}{210}$$

Since $\frac{q_{j|i}\left(q_{\hat{j}|(47-i)} + q_{\hat{j}|i}\right)}{210}$ must be an integer, and $q_{j|i}$ has no factors of 2, 3, 5, 7, $q_{\hat{j}|(47-i)} + q_{\hat{j}|i}$ must be a multiple of 210. This happens only when

$$q_{\hat{j}|(47-i)} + q_{\hat{j}|i} = \begin{cases} 210, \text{ if } \hat{j}\,|\,(47-i) + \hat{j}\,|\,i = 47 \\ 420, \text{ if } \hat{j}\,|\,(47-i) \text{ or } \hat{j}\,|\,i = 48 \end{cases}$$

Therefore,

$$l(47-i, j) + l(i, j) = \begin{cases} q_{j|i} + 1, \text{ if } i \neq j \text{ and } 47 - i \neq j \\ 2q_{j|i} + 1, \text{ if } i = j \text{ or } 47 - i = j \end{cases}$$

(ii): Notice that $r_{47} + r_{48} = 209 + 211 = 420$. Hence

$$l(47, j) + l(48, j) = \left(1 + \frac{q_{j|47} q_{\hat{j}|47} - r_{47}}{210}\right) + \left(1 + \frac{q_{j|48} q_{\hat{j}|48} - r_{48}}{210}\right)$$
$$= \frac{q_{j|i}\left(q_{\hat{j}|47} + q_{\hat{j}|48}\right)}{210}$$
$$= \begin{cases} q_{j|i}, \text{ if } \hat{j}\,|\,47 + \hat{j}\,|\,48 = 47 \\ 2q_{j|i}, \text{ if } \hat{j}\,|\,47 \text{ or } \hat{j}\,|\,48 = 48 \end{cases}$$

The 2nd case happens only when $j = 47, 48$.

7. Justification for Establishing $L_b(1)$ and Figure S2

Starting with $\theta = 1$, compute $q_j, q_{\hat{j}}$ and $l(1, j)$ using Expression (1)

$$l(1, j) = 1 + \frac{q_j \times q_{\hat{j}} - 11}{210}, \text{ for } j = 1, 2, \cdots, 48.$$

Then for $2 \le \theta \le 19$ and $\hat{\theta} = 1$, we have:

$$l(1, j + 48) = l(1, j) + q_{\hat{j}}$$

$$l(1, j+96) = l(1, j) + 2q_{\hat{j}}$$

$$\vdots$$

$$l(1, j+864) = l(1, j) + 18q_{\hat{j}}$$

We then list $l(i, j+48(\theta-1))$ for $\theta = 1,2,3,\cdots,19$ in **Figure S2**, corresponding to Expression (2'). The bottom row of **Figure S3** demonstrates that $\Delta_{1,1} = 2$, $\Delta_{1,2} = 15$, $\Delta_{1,3} = 28$, $\cdots$, $\Delta_{1,19} = 210$. Notably, $\min_j \{l(i, j+48(\theta-1))\} < \min_j \{l(i, j+48\theta)\}$ for all $\theta = 1,2,3,\cdots,19$, and $\Delta_{i,1} < \Delta_{i,2} < \Delta_{i,3} < \cdots$, for all *i*.

For instance, given $b = 2311 = 211 + 210 \times 10$, compute $L_{2311}(1)$ by applying Expression (2)to obtain $L_{2311}(1) = \{2,5,7,9\}$.

According to the Formula of Primes, in order for $\alpha = 11 + 210k \le 2311$ to be primes, it requires $k+1 \notin L_{2311}(1)$ for $k \in \{1,2,\cdots,10\}$. Therefore, we let $k \in \{0,2,3,5,7,9,10\}$ and get the following primes: 11, $11 + 210 \times 2 = 431$, $11 + 210 \times 3 = 641$, $11 + 210 \times 5 = 1061$, $11 + 210 \times 7 = 1481$, $11 + 210 \times 9 = 1901$ and $11 + 210 \times 10 = 2111$. They are all shown in the first row of **Figure 4**.

Given $b = 211 + 210 \times 28 = 6091$, we get

$$L_{6091}(1) = \{2,5,7,9,12,15,16,17,19,23,24,28\}.$$

In order for $\alpha = 11 + 210k \le 6091$ to be primes, we let $k \in \{0,2,3,5,7,9,10,12,13,17,19,20,21,24,25,26\}$ such that $k+1 \in L_{6091}(1)$ to obtain the following primes: 11, 431, 641, 1061, 1481, 1901, 2111, 2531, 2741, 3581, 4001, 4211, 4421, 5051, 5261, 5471.

8. Elaborations of Expressions (3) and (4)

Proof of Expression (3)

The total number of elements in PTP⁺(0, *k*) is $48 \times (k+1)$, for $k \in N_0$. Within the interval $[11, 211^2\theta]$, the number of different $l(i, j)$ is $|L_b(i)|$ for $b = 211 + 210k$. Hence, the total number of composites without 2, 3, 5, and 7 within the above interval is $|L_b(i)|$. Note that 4 primes are less than 10. Therefore, the number of primes within $[2, b]$ is $\pi(b) = 4 + 48 \times (k+1) - \sum_{i=1}^{48} |L_b(i)|$.

Proof of Expression (4)

For an integer $i \in W = \{1,3,6,9,13,\cdots,45,47\}$, (r_i, r_{i+1}) are twin primes for $r_i \ge 11$. If there exists a *k*, $k \in N_0$, such that $k+1 \notin L_b(i)$ and $k+1 \notin L_b(i+1)$, then $r_i + 210k$ and $r_{i+1} + 210k$ are twin primes. Since (2, 3), (3, 5), (5, 7) are the only three twin-prime pairs less than 11. Therefore, the number of twin-prime pair in interval $[2, b]$ is $\pi^*(b) = 3 + \sum_{i \in W} |T(i, i+1)|$, where $T(i, i+1)$ is a set of *k* such that $k+1 \notin L_b(i)$ and $k+1 \notin L_b(i+1)$.

9. Numerical Illustrations for The Periodic Table of Primes

Let $\lambda(i, k)$ be the integer value of the entry (i, k) in PTP⁺(0, 210) if **Figure 3**, which is computed based on CTC (1) and the Formula of Primes. For instance, $\lambda(i,k)$ on the diagonal dotted points (1, 11), (2, 13), (3, 17), (4, 19) and (5, 23) are computed below:

$l(1,1)=12$, $l(2,2)=14$, $\cdots$, $l(23,23)=104$, $\cdots$, $l(46,46)=200$ are converted to $\lambda(1,11)$, $\lambda(2,13)$, $\cdots$, $\lambda(23,17)$, $\cdots$, $\lambda(46,103)$, respectively, shown by red numbers as

$$\lambda(1,11)=r_1+210(l(1,1)-1)=11+210(12-1)=2321=11\times 211,$$

$$\lambda(2,13)=r_2+210(l(2,2)-1)=13+210(14-1)=2743=13\times 211,$$

$$\vdots$$

Similarly,

$$\lambda(23,103)=r_{23}+210(l(23,23)-1)=103+210(104-1)=21733=103\times 211$$

$$\lambda(46,199)=r_{46}+210(l(46,46)-1)=199+210(200-1)=41989=199\times 211$$

Also, $l(46,1)=11$, $l(45,2)=13$, $\cdots$, $l(1,46)=199$ are converted to $\lambda(i,k)$, respectively, as

$$\lambda(46,10)=r_{46}+210(l(46,1)-1)=199+210(11-1)=2299=11\times 209$$

$$\vdots$$

$$\lambda(1,198)=r_1+210(l(1,46)-1)=11+210(199-1)=41591=199\times 209$$

Now return to columns 22 and 25 in **Figure 2**. We note that (*i*, 23) and (*i*, 24), for $i=1,2,\cdots,48$ being the intermediate dotted points. There are two other green lines 12–10–9–7–6 for column 22, and 6–7–9–10–12 for column 25. Both are caused by the intermedia effect, with respect to 47. These two lines are also converted to **Figure 4**, shown as 11 × 101 – 13 × 101 – 17 × 101 – 19 × 101 – 23 × 101, and 11 × 109 – 13 × 109 – 17 × 109 – 19 × 109 – 23 × 109, respectively.

The above conversions between $l(i,j)$ in the CTC and $\lambda(i,k)$ in the PTP$^+$ demonstrate that (i) many composites such as the digonal and intermediate dotted points which allocate cyclically at the CTC are also allocated periodically at the PTP$^+$. (ii) the PTP is obtained by removing all composites periodically allocated at PTP$^+$. Therefore, we claim that prime numbers are also generated periodically since they come from removing all periodically distributed composites.

Figure 4 contains the first 11 columns for the PTP$^+$(0, 210), which is composed of 48 × 11 = 528 elements, represented by $\lambda(i,k)$ and elaborated below.

(i) Each entry $\lambda(i,k)$ is either a prime or a composite.

(ii) If $\lambda(i,k)$ is a prime, then $\lambda(i,k)=r_i+210k$, and vice versa. If $\lambda(i,k)$ is a composite, then there are q_j and $q_{\hat{j}}$ such that

$$\lambda(i,k)=q_j\times q_{\hat{j}}=r_i+210\times(l(i,j)-1).$$

Take $i=1$, $r_i=11$ for instance,

$$\lambda(1,1)=11+210\times(l(1,2)-1)=q_2\times q_3=13\times17,$$

$$\lambda(1,4)=11+210\times(l(1,5)-1)=q_5\times q_8=23\times37.$$

Then, we have $\lambda(1,0)=11$, $\lambda(1,2)=11\times210(3-1)=431$,

$$\lambda(1,3)=11+210\times(4-1)=641,\ \cdots,$$

$$\lambda(1,9)=11+210\times(10-1)=1901.$$

(iii) Primes $\lambda(i,k)$ and $\lambda'(i,k)=\lambda(i,k)+1$ are twin-prime pairs, if and only if $r_i'=2+r_i$, where $\lambda(i,k)=r_i+210\times k$, $\lambda'(i,k)=r_i'+210\times k$.

For instance, (431, 433), (641, 643), (1061, 1063), (1481, 1483) are twin-prime pairs.

(iv) The numbers of primes in column k of the PTP$^+$(0, 210) are shown as (43, 35, 32, 31, 31, 28, 28, 30, 26, 27, ⋯). The sum of primes within interval [11, 2311] is 43 + 35 + 32 + ⋯+ 18 = 329. The number of twin-prime pairs of a column k are listed as (12, 6, 4, 4, 4, 8, 6, 4, 4, 4, ⋯, 3).The maximum prime gap for the k^{th} column can also be found from **Figure 4**. For instance, the maximum prime gap for column 2 is 10, occurring at 409 and 419 of r_{46} and r_{47}, respectively.

(v) The PTP(k^0,k) is useful in predicting primes. Examples are given by Examples for the Predictions of Primes by the PTP.

In summary, the Formula of Primes, for the case $\hat{\theta}=1$, is checked below:

From **Figure S1(a)**, we know that

$$L_b(1)=\{2,5,7,9,12,\cdots\}$$

$$L_b(2)=\{7,10,12,13,14,\cdots\}$$

$$L_b(3)=\{3,6,11,12,13,14,\cdots\}$$

$$\vdots$$

From **Figure 4**, we know that

$$\alpha=r_1+210k \text{ is prime for } k=0,2,3,5,7,9,10$$

$$\alpha=r_2+210k \text{ is prime for } k=0,1,2,3,4,5,7,8,10$$

$$\alpha=r_3+210k \text{ is prime for } k=0,1,3,4,6,7,8,9$$

$$\vdots$$

In general, $\alpha=r_i+210k$ is prime, if $k+1\notin L_b(i)$ and $\hat{\theta}=1$.

10. Examples for the Predictions of Primes by the PTP

Example 1: Prediction of the next prime after 1,951

Since $1951=61+210(10-1)$, from **Figure 4** there are four consecutive composites, *i.e.* 19 × 103, 37 × 53, 13 × 151, and 11 × 179, right after 1951. We know the prime next to 1951 is 1973.

Example 2: Prediction of the 300th prime

From Expression (3), given b = 1891, the number of primes within [2, 1891] is counted as

$$\pi(1891) = 4 + 48 \times (8+1) - \sum_{i=1}^{48} \left| L_b(i) \right| = 4 + 432 - 146 = 290 .$$

The 290th prime is 1889, which is located at $(i,k) = (47,8)$ as shown in **Figure 4**. Therefore, the 300th prime is counted to be 1987, at the location of $(i,k) = (21,9)$.

Example 3: Prediction of the next twin primes after the twin primes pair (1277, 1279)

The location of 1277 is at $(i,k) = (3,6)$. The twin primes $\lambda(i,k)$ and $\lambda' = \lambda(i,k) + 1$ are expressed as $\lambda(i,k) = r_i + 210(7-1)$. $\lambda' = r_i' + 210(7-1)$ and $r_i' = r_i + 2$.
Therefore, $r_i = 29$, $r_i' = 31$, and $\lambda(i,k) = 1289$, $\lambda' = 1291$.

Example 4: Predicting the longest same-difference primes less than or equal to b

The same-difference in this example means 210.
We can predict the longest same-difference prime series in PTP+(0, 10) from the CTC (basic) by checking all rows of **Figure S1(a)**.
For each row i, we find $L_{2311}(i), i = 1,2,3,\cdots,48$, below:

$L_{2311}(1) = \{2,5,7,9\}$, $L_{2311}(2) = \{10,7\}$, $L_{2311}(3) = \{11,3,6\}$,
$L_{2311}(4) = \{10,11,4\}$, $L_{2311}(5) = \{11,6,9\}$, $L_{2311}(6) = \{10,5,6\}$,
$L_{2311}(7) = \{10,3,5,6,8,9\}$, $L_{2311}(8) = \{10,2,4,8,9\}$, $L_{2311}(9) = \{4\}$,
$L_{2311}(10) = \{2,8\}$, $L_{2311}(11) = \{10,11,8,9\}$, $L_{2311}(12) = \{10,3,5,7\}$,
$L_{2311}(13) = \{11,4,5,8,9\}$, $L_{2311}(14) = \{3,5,6\}$, $L_{2311}(15) = \{10,11,4,8\}$,
$L_{2311}(16) = \{10,11,6,7,8,9\}$, $L_{2311}(17) = \{10,11,3,4,5,7\}$,
$L_{2311}(18) = \{10,2,7\}$,
$L_{2311}(19) = \{11,4,5,6,7,9\}$, $L_{2311}(20) = \{11,2,6,7,9\}$,
$L_{2311}(21) = \{11,3,6,7\}$,
$L_{2311}(22) = \{10,11,4,9\}$, $L_{2311}(23) = \{5,7,8\}$, $L_{2311}(24) = \{3,4,6,8\}$,
$L_{2311}(25) = \{11,2,3,5,6,7\}$, $L_{2311}(26) = \{2,3,9\}$, $L_{2311}(27) = \{1,5,8\}$,
$L_{2311}(28) = \{11,6,7,9\}$, $L_{2311}(29) = \{10,11,2,3,7\}$, $L_{2311}(30) = \{4,7,9\}$,
$L_{2311}(31) = \{3,5,6,9\}$, $L_{2311}(32) = \{1,10,7\}$, $L_{2311}(33) = \{11,4,5,6,9\}$,
$L_{2311}(34) = \{10,2,4,7\}$, $L_{2311}(35) = \{10,11,6,7,9\}$,
$L_{2311}(36) = \{11,3,4,5,8,9\}$,
$L_{2311}(37) = \{10,2,5\}$, $L_{2311}(38) = \{1,10,3,4,6,8,9\}$, $L_{2311}(39) = \{4,8,9\}$,
$L_{2311}(40) = \{11,8,9\}$, $L_{2311}(41) = \{10,2,7,8\}$, $L_{2311}(42) = \{1,10,4,5\}$,
$L_{2311}(43) = \{11,3,6,8\}$, $L_{2311}(44) = \{2,6\}$, $L_{2311}(45) = \{2,5,6,7\}$,
$L_{2311}(46) = \{11\}$, $L_{2311}(47) = \{1,3,7,8\}$, $L_{2311}(48) = \{10,4,6,8,9\}$.

Apparently, the longest same-difference primes is indicated by $L_{2311}(46)=\{11\}$.
Therefore, α is prime, for $\alpha = 199 + 210k$, $k=\{0,1,2,3,4,\cdots,9\}$.

We then find the longest same-difference prime series in PTP⁺ [0, 10], or a 10-tuple primes series, to be (199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089), as shown on row 46 of **Figure 4**.

Similarly, we can find the next longest same-difference prime series in PTP⁺ (0, 10), which is associated with $L_{2311}(9)=\{4\}$. Namely, it happens at row 9, listed to be (881, 1091, 1301, 1511, 1721, 1931, 2141), as shown on row 9 of **Figure 4**.

Example 5: How many primes are within the interval [2, 2311]?
From the CTCs, we count to find $\sum_{i=1}^{48}|L_b(i)|=188$ for $b=2311$.
From Expression (3), $\pi(2311)=4+48(10+1)-188=344$.

Example 6: How many twin-prime pairs are within the interval [11, 2311]?
Referring to Expression (4), let $b=2311$, from CTC, we compute $T(i,i+1)$, for $i\in\{1,3,6,9,13,16,22,24,30,33,37,40,43,45,47\}$ to reach

$$T(1,2)=\{k \mid k+1\notin L_b(1), k+1\notin L_b(2)\}=\{0,2,3,5,7,10\}$$

Similarly, we have

$$T(3,4)=\{0,1,4,6,7,8\},\quad T(6,7)=\{0,1,3,6\}, \text{ and}$$

$$T(9,10)=\{0,2,4,5,6,8,9\}, \text{ and } \quad T(47,48)=\{0,1,4,10\}.$$

Therefore, the number of twin-prime pairs is

$$\begin{aligned}\pi^*(2311) &= 3+\sum_{i\in w}|T(i,i+1)| \\ &= |T(1,2)|+|T(3,4)|+|T(6,7)|+\cdots+|T(47,48)|+3 \\ &= 6+6+\cdots+4+3=68.\end{aligned}$$

root i \ r_i \ $l(i,j)$ \ segment q_j \ j		1	2	3	4	5	6	7	8	9	10	11	12	···	23	24	···	46	47	48
		11	13	17	19	23	29	31	37	41	43	47	53	···	103	107	···	199	209	211
1	11	12	2	2	9	5	16	7	5	7	29	24	33		24	84			199	12
2	13	10	14	13	7	12	24	25	36	23	38	41	34		75	31			197	14
3	17	6	12	18	3	3	11	30	24	14	13	28	36		74	32			193	18
4	19	4	11	12	20	10	19	17	18	30	22	45	37		22	86			191	20
5	23	11	9	17	16	24	6	22	6	21	40	32	39		21	87			187	24
6	29	5	6	16	10	22	30	14	25	28	24	36	42		71	35			181	30
7	31	3	5	10	8	6	9	32	19	3	33	6	43		19	89			179	32
8	37	8	2	9	2	4	4	24	38	10	17	10	46		69	37			173	38
9	41	4	13	14	17	18	20	29	26	42	35	44	48		68	38			169	42
10	43	2	12	8	15	2	28	16	20	17	44	14	49		16	92			167	44
11	47	9	10	13	11	16	15	21	8	8	19	48	52		15	93			163	48
12	53	3	7	12	5	14	10	13	27	15	3	5	54		65	41			157	54
13	59	8	4	11	18	12	5	5	9	22	30	9	4		12	96			151	60
14	61	6	3	10	16	19	13	23	3	38	39	26	5		63	43			149	62
15	67	11	13	4	10	17	8	15	22	4	23	30	8		10	98			143	68
16	71	7	11	9	6	8	24	20	10	36	41	17	10		9	99			139	72
17	73	5	10	3	4	15	3	7	4	11	7	34	11		60	46			137	74
18	79	10	7	2	17	13	27	30	23	18	34	38	14		7	101			131	80
19	83	6	5	7	13	4	14	4	11	9	9	25	16		6	102			127	84
20	89	11	2	6	7	2	9	27	30	16	36	29	19		56	50			121	90
21	97	3	11	16	18	7	12	6	6	39	29	3	23		54	52			113	98
22	101	10	9	4	14	21	28	11	31	30	4	37	25		53	53			109	102
23	103	8	8	15	12	5	7	29	25	5	13	7	26		104	107			107	104
47	209	1	7	3	1	8	25	22	3	33	17	15	26		26	80			210	210

Note: Numbers in red — for diagonal effect. Numbers in green — for intermediate effect.

(a)

root i \ r_i \ $l(i,j)$ \ segment q_j \ j		1	2	3	4	5	6	7	8	9	10	11	12	···	23	24	···	46	47	48
		11	13	17	19	23	29	31	37	41	43	47	53	···	103	107	···	199	209	211
24	107	4	6	3	8	19	23	3	13	37	31	41	28		103	108			103	108
25	109	2	5	14	6	3	2	21	7	12	40	11	29		51	55			101	110
26	113	9	3	2	2	17	18	26	32	3	15	45	31		50	56			97	114
27	121	1	12	12	13	22	21	5	8	26	8	19	35		48	58			89	122
28	127	6	9	11	7	20	16	28	27	33	35	23	38		98	6			83	128
29	131	2	7	16	3	11	3	2	15	24	10	10	40		97	7			79	132
30	137	7	4	15	16	9	27	25	34	31	37	14	43		44	62			73	138
31	139	5	3	9	14	16	6	12	28	6	3	31	44		95	9			71	140
32	143	1	1	14	10	7	22	17	16	38	21	18	46		94	10			67	144
33	149	6	11	13	4	5	17	9	35	4	5	22	49		41	65			61	150
34	151	4	10	7	2	12	25	27	29	20	14	39	50		92	12			59	152
35	157	9	7	6	15	10	20	19	11	27	41	43	53		39	67			53	158
36	163	3	4	5	9	8	15	11	30	34	25	47	3		89	15			47	164
37	167	10	2	10	5	22	2	16	18	25	43	34	5		88	16			43	168
38	169	8	1	4	3	6	10	3	12	41	9	4	6		36	70			41	170
39	173	4	12	9	18	20	26	8	37	32	27	38	8		35	71			37	174
40	179	9	9	8	12	18	21	31	19	39	11	42	11		85	19			31	180
41	181	7	8	2	10	2	29	18	13	14	20	12	12		33	73			29	182
42	187	1	5	1	4	23	24	10	32	21	4	16	15		83	21			23	188
43	191	8	3	6	19	14	11	15	20	12	22	3	17		82	22			19	192
44	193	6	2	17	17	21	11	2	14	28	31	20	18		30	76			17	194
45	197	2	13	10	13	12	6	7	2	19	6	7	20		29	77			13	198
46	199	11	12	16	11	19	14	25	33	35	15	24	21		80	24			11	200
48	211	10	6	14	18	15	4	9	34	8	26	32	27		77	27			208	212

Note: Numbers in red — for diagonal effect. Numbers in green — for intermediate effect.

(b)

Figure S1. (a): A basic composite table CTC (basic); (b): A complementary composite table CTC (complementary).

j	θ = 1			θ = 2	θ = 3		θ = 19
	q_j	$q_{\bar{j}}$	$l(1,j)$	$l(1, j+48) = l(1, j)+q_j$	$l(1, j+96) = l(1, j)+2q_j$		$l(1, j+864) = l(1, j)+18q_j$
1	11	211	12	223	434		3810
2	13	17	2	19	36		308
3	17	13	2	15	28		236
4	19	89	9	98	187		1611
5	23	37	5	42	79		671
6	29	109	16	125	234		1978
7	31	41	7	48	89		745
8	37	23	5	28	51		419
9	41	31	7	38	69		565
10	43	137	29	166	303		2495
11	47	103	24	127	230		1878
12	53	127	33	160	287		2319
13	59	139	40	179	318		2542
14	61	131	39	170	301		2397
15	67	113	37	150	263		2071
16	71	151	52	203	354		2770
17	73	167	59	226	393		3065
18	79	149	57	206	355		2739
19	83	157	63	220	377		2889
20	89	19	9	28	47		351
21	97	143	67	210	353		2641
22	101	181	88	269	450		3346
23	103	47	24	71	118		870
24	107	163	84	247	410	…	3018
25	109	29	16	45	74		502
26	113	67	37	104	171		1243
27	121	191	111	302	493		3549
28	127	53	33	86	139		987
29	131	61	39	100	161		1137
30	137	43	29	72	115		803
31	139	59	40	99	158		1102
32	143	37	67	104	141		733
33	149	79	57	136	215		1479
34	151	71	52	123	194		1330
35	157	83	63	146	229		1557
36	163	107	84	191	298		2010
37	167	73	59	132	205		1373
38	169	179	145	324	503		3367
39	173	187	155	342	529		3521
40	179	169	145	314	483		3187
41	181	101	88	189	290		1906
42	187	173	155	328	501		3269
43	191	121	111	232	353		2289
44	193	197	182	379	576		3728
45	197	193	182	375	568		3656
46	199	209	199	408	617		3961
47	209	199	199	398	597		3781
48	211	11	12	23	34		210
$\Delta_{i,\theta}$			2	15	28		210

$$\Delta_{i,\theta} = \min_j \{l(i, j+48(\theta-1))\}, i=1$$

Figure S2. List of $L_{44521}(1)$, corresponding to Expression (2’), for $\theta = 1, 2, \ldots, 19$.

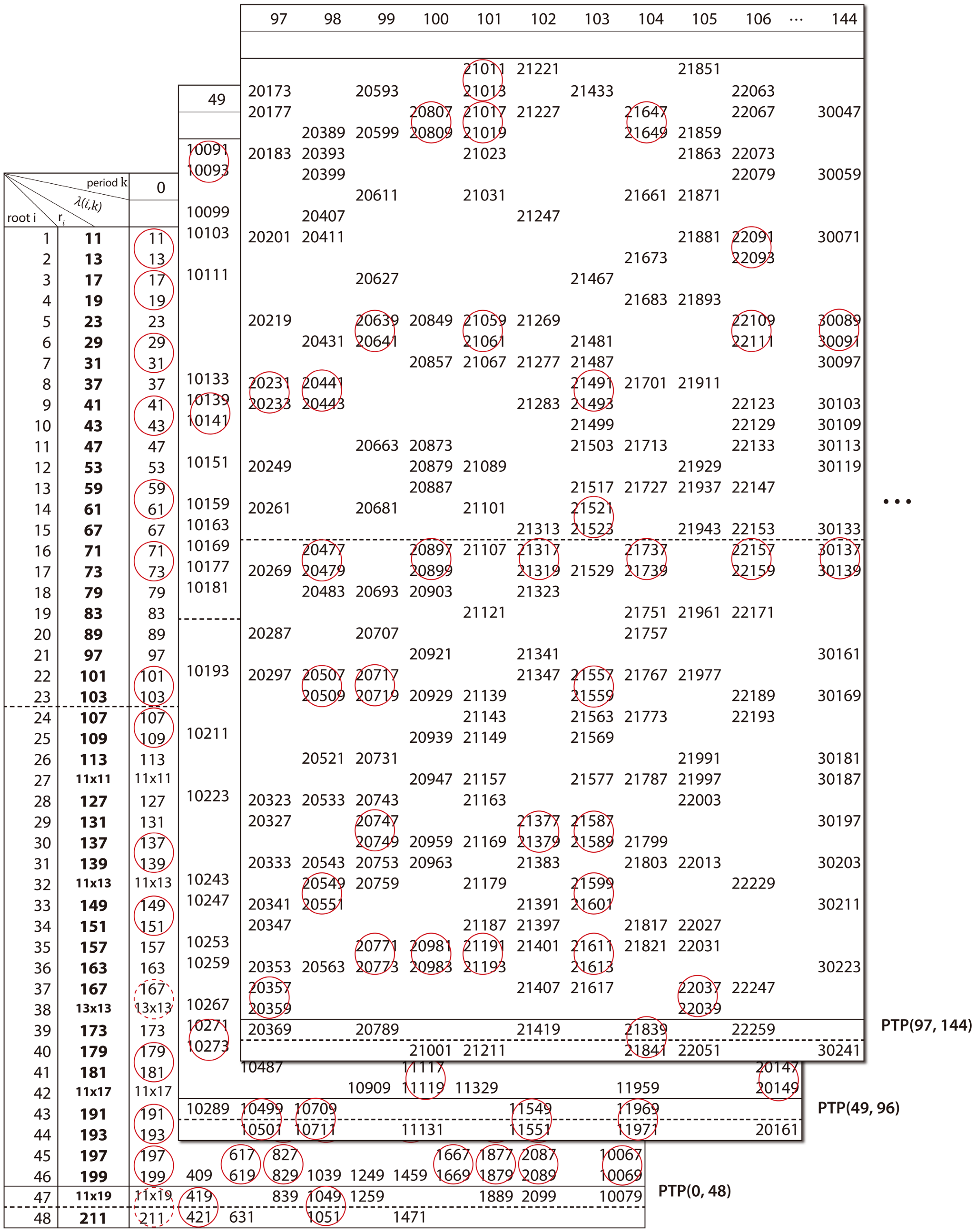

Figure S3. The periodic table of primes for various intervals.