Asian Research Journal of Mathematics

Volume 20, Issue 11, Page 26-41, 2024; Article no.ARJOM.124434
ISSN: 2456-477X

Enhanced Adomian Decomposition Method for Accurate Numerical Solutions of PDE Systems

Saleh Ali Alomari [a*] and Yahya Qaid Hasan [b]

[a] *Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Yemen.*
[b] *Department of Mathematics, Faculty of Applied Science, Thamar University, Yemen.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: https://doi.org/10.9734/arjom/2024/v20i11857

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/124434

Received: 07/08/2024
Accepted: 14/10/2024
Published: 30/10/2024

Original Research Article

Abstract

This research addresses critical challenges in numerical solutions, which are vital for various engineering and physical fields. The Modified Adomian Decomposition Method (MADM) is proposed as a novel approach for solving linear and nonlinear partial differential equations (PDEs). MADM builds upon the Adomian Decomposition Method (ADM) by incorporating a new integral operator that significantly improves convergence rates and accuracy.

**Corresponding author: E-mail: salehory2020@gmail.com;*

Cite as: *Alomari, Saleh Ali, and Yahya Qaid Hasan. 2024. "Enhanced Adomian Decomposition Method for Accurate Numerical Solutions of PDE Systems". Asian Research Journal of Mathematics 20 (11):26-41. https://doi.org/10.9734/arjom/2024/v20i11857.*

Numerical examples demonstrate the effectiveness of MADM in handling complex nonlinear PDEs. Compared to traditional ADM, MADM consistently achieves more accurate and rapidly converging solutions. This enhancement is attributed to the novel integral operator, which addresses the limitations of ADM for intricate nonlinear problems.
The paper outlines the application of MADM, its solution procedure, and its effectiveness through numerical examples. Comparisons with standard ADM solutions and exact solutions validate MADM's accuracy and superiority. The results suggest that MADM is a promising tool for expanding the applicability of Adomian methods in the field of solving PDEs.

Keywords: Adomian decomposition method; system of PDEs; numerical methods; accuracy; convergence; approximate solutions.

1 Introduction

Partial Differential Equations (PDEs) are indispensable tools for modeling diverse scientific and engineering phenomena. Accurately solving these equations is vital for predicting system behavior, optimizing designs, and analyzing experimental data (Wazwaz, 2000; Wazwaz, 2007; Loyinmi, 2024). Traditional methods, however, may not be suitable for all PDEs, especially nonlinear ones. This has led to the exploration of alternative approaches (Loyinmi, 2024), such as the well-established Adomian Decomposition Method (ADM).

ADM addresses nonlinear PDEs by decomposing the solution into an infinite series (Adomian, 1990; Sekgothe, 2021). Researchers have continuously explored modifications and extensions to ADM (Adomian and Rach, 1992; Aland and Singh, 2022; Babolian et al., 2004; Daftardar-Gejji and Jafari, 2005; Hasan, 2014; Kanth and Aruna, 2008; Kaya and El-Sayed, 2004), broadening its applicability to various types of differential equations (Fadaei, 2011; Loufouilou et al., 2021; Nuruddeen et al., 2018; Almardy et al., 2023; Alomari and Hasan, 2023) Recent advancements, including the integration of integral transforms, have significantly enhanced ADM's capability to solve complex, higher-order, and fractional differential systems (Dubey et al., 2020; Rab et al., 2024; Alsidrani et al., 2024; Khan et al., 2020).

This research introduces a novel approach, the Modified Adomian Decomposition Method (MADM), which builds upon ADM. MADM incorporates a new integral operator specifically (Hasan, 2012; AL-Mazmumy et al., 2022) designed to improve convergence and accuracy for complex nonlinear problems. Numerical experiments demonstrate MADM's superior performance in obtaining approximate solutions compared to traditional methods. The potential applications of MADM in fields such as fluid dynamics and heat transfer highlight its significance as a valuable tool for computational mathematics.

The following sections provide a detailed explanation of ADM and MADM, including their application to specific nonlinear PDEs. Numerical examples are presented to showcase MADM's effectiveness in handling nonlinearities and converging towards approximate solutions. The conclusion emphasizes the potential benefits of MADM in solving complex nonlinear problems.

2 ADM and MADM for PDE System Solutions

2.1 Analysis of ADM

The system of partial differential equations that will be studied in this section takes the general form of:

$$u_t^i(x,t) + \lambda_i(t)u_i(x,t) + N_i(u_1,\ldots,u_m) = g_i(x), \quad i = 1,2,\ldots,m. \tag{2.1}$$

The initial conditions are given by

$$u_i(x,0) = f_i(x).$$

The equation (2.1) is equivalent to

$$L_i u_i(x,t) + \lambda_i(t) u_i(x,t) + N_i(u_1,\ldots,u_m) = g_i(x), \quad i = 1,2,\ldots,m. \tag{2.2}$$

where $L_i = \frac{d}{dt}$ represents partial differential operators, R_i and N_i represent linear and nonlinear operator respectively.

Applying the inverse operators $L_i^{-1}() = \int_0^t () \ dt$ to system (2.2) and applying beginning conditions, then

$$u_i(x,t) = f_i(x) + L_i^{-1} g_i(x) - L_i^{-1} \lambda_i(t) u_i(x,t) - L_i^{-1} N_i(u_1,\ldots,u_m), \quad i = 1,2,\ldots m. \tag{2.3}$$

The Adomian decomposition method approach presupposes that the unknown functions $u_i(x,t)$ can be written as an infinite series of the form

$$u_i(x,t) = \sum_{n=0}^{\infty} u_n^i(x,t), \quad i = 1,2,\ldots,m. \tag{2.4}$$

The nonlinear operator $N_i(u_1,\cdots,u_m)$ is defined using Adomian polynomials.

$$N_i(u_1,\ldots,u_m) = \sum_{n=0}^{\infty} A_n^i, \quad i = 1,2,\ldots,m, \tag{2.5}$$

where

$$A_n^i = \frac{1}{n!}\left[\frac{d^n}{d\mu^n} N_i\left(\sum_{n=0}^{\infty} u_{1n}\mu^n,\ldots,\sum_{n=0}^{\infty} u_{mn}\mu^n\right)\right]_{\mu=0}, \quad n \geq 0,$$

is an Adomian polynomial that can be constructed for all forms of all nonlinearity.

By (2.4) and (2.5) in (2.3) yields:

$$\sum_{n=0}^{\infty} u_n^i(x,t) = f_i(x) + L_i^{-1} g_i(x) - L_i^{-1}\left[\lambda_i(t)\left(\sum_{n=0}^{\infty} u_n^1,\ldots,\sum_{n=0}^{\infty} u_n^m\right)\right] - L_i^{-1}\left(\sum_{n=0}^{\infty} A_n^i\right). \tag{2.6}$$

Identifying the Zeroth component u_o^i for all terms. The recurrence relations can be used calculate the remaining components f_i and g_i, u_n^i, $n \geq 0$ can be determined by using the recurrence relations:

$$\begin{aligned} u_0^i &= f_i(x) + L_i^{-1} g_i(x), \\ u_{n+1}^i &= -L_i^{-1}\left[\lambda_i(t)\left(\sum_{n=0}^{\infty} u_n^1,\ldots,\sum_{n=0}^{\infty} u_n^m\right)\right] - L_i^{-1}\left(\sum_{n=0}^{\infty} A_n^i\right), \quad i = 1,2,\ldots,m, \quad n \geq 0. \end{aligned}$$

2.2 Analysis of MADM

In order to solve a system (2.1) that matched the following conditions. We can rewrite the system (2.1) as

$$u_t^i(x,t) + \lambda_i(t) u_i(x,t) = g_i(x) - N_i(u_1,\ldots,u_m), \quad i = 1,2,\ldots,m. \tag{2.7}$$

We define the direct operator and its corresponding inverse operator as follows

$L_i = e^{-\int \lambda_i(t) \ dt} \frac{d}{dt} e^{\int \lambda_i(t) \ dt}$, $L_i^{-1}() = e^{-\int \lambda_i(t) \ dt} \int_0^t e^{\int \lambda_i(t) \ dt}() \ dt$

Therefore

$$L_i u_i(x,t) = g_i(x) - N_i(u_1,\ldots,u_m), \quad i = 1,2,\ldots,m. \tag{2.8}$$

Taking the operator L_i^{-1} for $u_t^i + \lambda_i(t) u_i$ in the equation (2.7), we get

$$\begin{aligned} L_i^{-1}(u_t^i(x,t) + \lambda_i(t) u_i(x,t)) &= e^{-\int \lambda_i(t) \ dt} \int_0^t e^{\int \lambda_i(t) \ dt}(u_i(x,t) + \lambda_i(t) u_i(x,t)) dt \\ &= u_i(x,t) - u_i(x,0)\Phi(0)\Phi^{-1}(t), \end{aligned}$$

where $\Phi^{-1}(t) = e^{-\int \lambda_i(t)dt}$. Applying the inverse operators to the equation (2.8),we get

$$u_i(x,t) = u_i(x,0)\Phi(0)e^{-\int \lambda_i(t)dt} + L_i^{-1}g_i(x) - L_i^{-1}(N_i(u_1,\ldots,u_m)). \tag{2.9}$$

The recursive relation is identified by:

$$\begin{aligned} u_0^i &= f_i(x)\Phi(0)\Phi^{-1}(t) + L_i^{-1}g_i(x), \\ u_{n+1}^i &= -L_i^{-1}\left(\sum_{n=0}^{\infty} A_n^i\right) \quad i = 1,2\ldots,m. \end{aligned}$$

3 Numerical Examples

Example 3.1. *Consider the system of nonlinear PDEs*

$$\begin{aligned} u_t + v_x w_y - v_y w_x + u = 0, \\ v_t + w_x u_y + w_y u_x - v = 0, \\ w_t + u_x v_y + u_y v_x - w = 0. \end{aligned} \tag{3.1}$$

With initial conditions

$$\begin{aligned} u(x,y,0) = e^{x+y}, \\ v(x,y,0) = e^{x-y}, \\ w(x,y,0) = e^{-x+y}. \end{aligned}$$

To solve the system by the proposed method, we well be using the following direct and inverse operators

$L_1 = e^{-\int dt}\frac{d}{dt}e^{\int dt}$, $L_2 = e^{\int dt}\frac{d}{dt}e^{-\int dt}$, $L_3 = e^{\int dt}\frac{d}{dt}e^{-\int dt}$,

$L_1^{-1}() = e^{-\int dt}\int_0^t e^{\int dt}()dt$, $L_2^{-1}() = e^{\int dt}\int_0^t e^{-\int dt}()dt$, $L_3^{-1}() = e^{\int dt}\int_0^t e^{-\int dt}()\ dt$.

So we can write the equation (3.1) in the form

$$\begin{aligned} L_1(u) = -(v_x w_y - v_y w_x), \\ L_2(v) = -(w_x u_y + w_y u_x), \\ L_3(w) = -(u_x v_y + u_y v_x). \end{aligned} \tag{3.2}$$

Taking the operator L_1^{-1}, L_2^{-1} and L_3^{-1} for $u_t + u$, $v_t - v$, $w_t - w$ of the system (3.1) respectively, we obtain

$$\begin{aligned} L_1^{-1}(u_t + u) &= e^{-t}\int_0^t e^t(u_t + u)dt \\ &= u(x,y,t) - e^{x+y-t}, \\ L_2^{-1}(v_t - v) &= e^t\int_0^t e^{-t}(v_t - v)dt \\ &= v(x,y,t) - e^{x-y+t}, \\ L_3^{-1}(w_t - w) &= e^t\int_0^t e^{-t}(w_t - w)dt \\ &= w(x,y,t) - e^{-x+y+t}. \end{aligned} \tag{3.3}$$

Taking the operator L_1^{-1}, L_2^{-1} and L_3^{-1} for equations of the system (3.2) respectively, we obtain

$$\begin{aligned} u(x,y,t) = e^{x+y-t} - L_1^{-1}(v_x w_y - v_y w_x), \\ v(x,y,t) = e^{x-y+t} - L_2^{-1}(w_x u_y + w_y u_x), \\ w(x,y,t) = e^{-x+y+t} - L_3^{-1}(u_x v_y + u_y v_x). \end{aligned} \tag{3.4}$$

Using the formulae $u=\sum_{n=0}^{\infty}u_n$, $v=\sum_{n=0}^{\infty}v_n$ *and* $w=\sum_{n=0}^{\infty}w_n$ *in equation (3.4), yields*

$$\begin{aligned}\sum_{n=0}^{\infty}u_n &= e^{x+y-t}-L_1^{-1}\left(\sum_{n=0}^{\infty}A_n\right),\\ \sum_{n=0}^{\infty}v_n &= e^{x-y+t}-L_2^{-1}\left(\sum_{n=0}^{\infty}B_n\right),\\ \sum_{n=0}^{\infty}w_n &= e^{-x+y+t}-L_3^{-1}\left(\sum_{n=0}^{\infty}C_n\right),\end{aligned}\tag{3.5}$$

where A_n, B_n, C_n *are the polynomials for the nonlinear expression*

$(v_xw_y-v_yw_x)$, $(w_xu_y+w_yu_x)$, $(u_xv_y+u_yv_x)$ *respectively.*

The general solution of the system is as follows

$$\begin{aligned}u_0(x,y,t) &= e^{x+y-t},\\ u_{k+1}(x,t) &= -L_1^{-1}A_k=0,\quad k\geq 0,\\ v_0(x,y,t) &= e^{x-y+t},\\ v_{k+1}(x,t) &= -L_2^{-1}B_k=0,\quad k\geq 0,\\ w_0(x,y,t) &= e^{-x+y+t},\\ w_{k+1}(x,y,t) &= -L_3^{-1}C_k=0,\quad k\geq 0.\end{aligned}\tag{3.6}$$

The exact solution of the system is given by

$(u,v,w)=u_0(x,y,t),v_0(x,y,t),w_0(x,y,t)=(e^{x+y-t},e^{x-y+t},e^{-x+y+t})$.

That is closed solution (Ebiwareme, 2022; Wazwaz, 2000; Yavuz, 2019).

Example 3.2. *Consider system of linear partial differential equations of the following form:*

$$\begin{aligned}u_t-v_x-u+v &= -2,\\ v_t-u_x-u+v &= -2.\end{aligned}\tag{3.7}$$

With initial conditions:

$$\begin{aligned}u(x,t) &= 1+e^x,\\ v(x,t) &= -1+e^x.\end{aligned}$$

The exact solution is as follows (Loufouilou et al., 2021).

$$u(x,t)=1+e^{x+t},v(x,t)=-1+e^{x+t},$$

Let $L_1=e^{\int dt}\frac{d}{dt}e^{-\int dt}$, $L_2=e^{-\int dt}\frac{d}{dt}e^{\int dt}$,

so $L_1^{-1}()=e^{\int dt}\int_0^t e^{-\int dt}()dt$, $L_2^{-1}()=e^{-\int dt}\int_0^t e^{\int dt}()dt$.

We can be written Eq.(3.7) as

$$\begin{aligned}L_1(u) &= -2+v_x-v,\\ L_2(v) &= -2+u_x+u.\end{aligned}\tag{3.8}$$

Taking the operator L_1^{-1}, L_2^{-1} for $u_t - u$, $v_t + v$ of the system (3.7) respectively, we obtain

$$\begin{aligned} L_1^{-1}(u_t - u) &= e^t \int_0^t e^{-t}(u_t - u)dt \\ &= u - (e^t + e^{x+t}), \\ L_2^{-1}(v_t + v) &= e^{-t} \int_0^t e^t(v_t + v)dt \\ &= v - (-e^{-t} + e^{x-t}). \end{aligned} \tag{3.9}$$

Taking the operator L_1^{-1}, L_2^{-1} for equations of the system (3.8) respectively, we obtain

$$\begin{aligned} u(x,t) &= e^t + e^{x+t} + L_1^{-1}(-2) + L_1^{-1}(v_x - v), \\ v(x,t) &= -e^{-t} + e^{x-t} + L_2^{-1}(-2) + L_2^{-1}(u_x + u). \end{aligned} \tag{3.10}$$

The general solution of the system is as follows

$$\begin{aligned} u_0(x,t) &= e^t + e^{x+t} + L_1^{-1}(-2) \\ u_{k+1}(x,t) &= L_1^{-1}(v_{k_x} - v_k), \quad k \geq 0, \\ v_0(x,t) &= -e^{-t} + e^{x-t} + L_2^{-1}(-2), \\ v_{k+1}(x,t) &= L_2^{-1}(u_{k_x} + u_k), \quad k \geq 0, \end{aligned} \tag{3.11}$$

The first three solutions iterations are as follows

$$\begin{aligned} u_0 &= e^{x+t} - e^t + 2, \\ v_0 &= e^{x-t} + e^{-t} - 2, \\ u_1 &= \frac{3}{2}e^t + \frac{1}{2}e^{-t} - 2, \\ v_1 &= -\frac{3}{2}e^{-t} - \frac{1}{2}e^t + 2, \\ u_2 &= \frac{1}{2}te^t - \frac{5}{4}e^t - \frac{3}{4}e^{-t} + 2, \\ v_2 &= \frac{1}{2}te^{-t} + \frac{5}{4}e^{-t} + \frac{3}{4}e^t - 2. \\ &\vdots \end{aligned} \tag{3.12}$$

Summing the above iterations yields

$$\begin{aligned} u(x,t) = u_0 + u_1 + u_2 + \cdots = e^{x+t} - \frac{1}{4}e^{-t} + \frac{1}{2}te^t - \frac{3}{4}e^t + 2 + \cdots \\ v(x,t) = v_0 + v_1 + v_2 + \cdots = e^{x-t} + \frac{1}{4}e^t + \frac{1}{2}te^{-t} + \frac{3}{4}e^{-t} - 2 + \cdots \end{aligned} \tag{3.13}$$

Which are the series of the approximate solution for $u(x,t)$and $v(x,t)$ given using MADM.

And

The series of the approximate solution for $u(x,t)$and $v(x,t)$ which are given using ADM as

$$\begin{aligned} u(x,t) = u_0 + u_1 + u_2 + \cdots = 1 + e^x + te^x + \frac{1}{2}t^2e^x + \cdots \\ v(x,t) = v_0 + v_1 + v_2 + \cdots = -1 + e^x + te^x + \frac{1}{2}t^2e^x + \cdots \end{aligned} \tag{3.14}$$

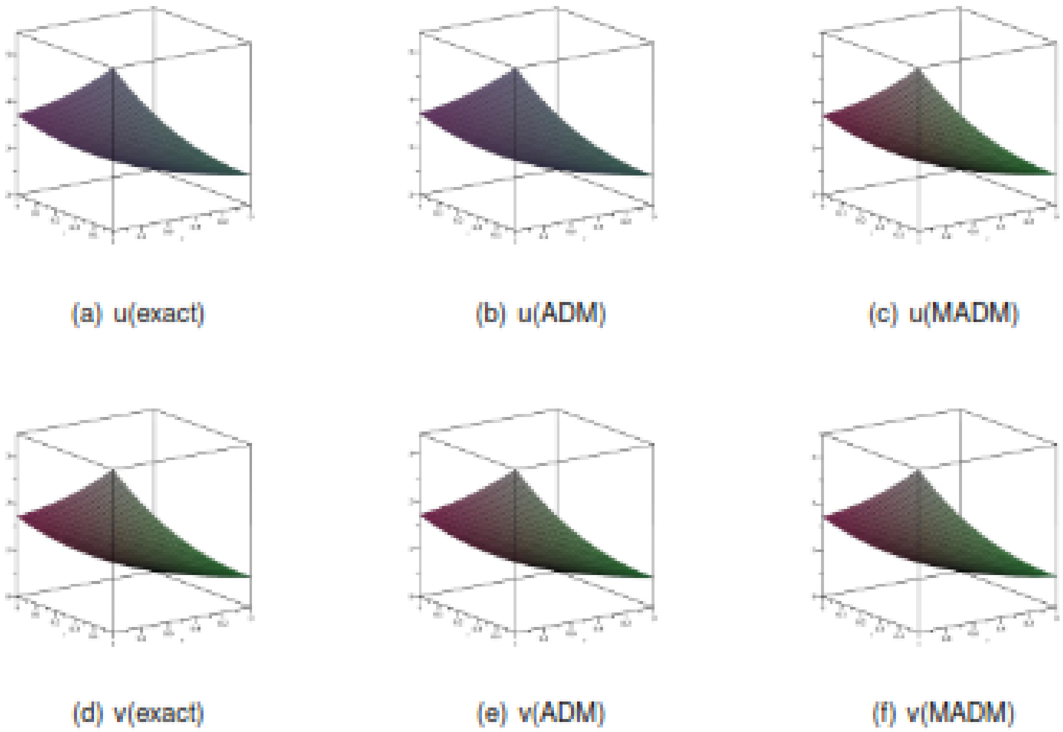


Fig. 1. 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM for Example(3.2)

Example 3.3. *Consider the coupled system of nonlinear partial differential equations of the following form:*

$$\begin{aligned} u_t + wu_x - \alpha u &= \beta, \\ w_t - uw_x + \alpha w &= \beta. \end{aligned} \tag{3.15}$$

With initial conditions:

$$\begin{aligned} u(x,t) &= e^{\beta x}, \\ w(x,t) &= e^{-\beta x}. \end{aligned}$$

The exact solution is as follows ((Ebiwareme, 2022),(Wazwaz, 2000)).

$$\begin{aligned} u(x,t) &= e^{\beta x + \alpha t}, \\ w(x,t) &= e^{-\beta x - \alpha t}, \end{aligned}$$

where α and β are constants.

To utilize the proposed technique, we start with this system representation:

$$\begin{aligned} L_1 u &= \beta - wu_x, \\ L_2 w &= \beta + uw_x, \end{aligned} \tag{3.16}$$

where $L_1 = e^{\int \alpha dt}\frac{d}{dt}e^{\int -\alpha dt}$, $L_2 = e^{-\int \alpha\ dt}\frac{d}{dt}e^{\int \alpha dt}$,
so $L_1^{-1}() = e^{\int \alpha\ dt}\int_0^t e^{\int -\alpha\ dt}()dt$, $L_2^{-1}() = e^{-\int \alpha\ dt}\int_0^t e^{\int \alpha\ dt}()\ dt$

Using the inverse integration operator L_1^{-1}, L_2^{-1} for $u_t - \alpha u$, $w_t + \alpha w$ of the system (3.15) respectively, possible to get

$$\begin{aligned} L_1^{-1}(u_t - \alpha u) &= e^{\alpha t} \int_0^t e^{-\alpha t}(u_t - \alpha u)\ dt = u - e^{\beta x + \alpha t}, \\ L_2^{-1}(w_t + \alpha w) &= e^{-\alpha t} \int_0^t e^{\alpha t}(w_t + \alpha w)\ dt = w - e^{-\beta x - \alpha t}. \end{aligned} \tag{3.17}$$

Using the inverse integration operator L_1^{-1}, L_2^{-1} for the equations of the system (3.16) respectively, we get

$$\begin{aligned} u(x,t) &= e^{\beta x + \alpha t} + L_1^{-1}(\beta) - L_1^{-1}(w u_x), \\ w(x,t) &= e^{-\beta x - \alpha t} + L_2^{-1}(\beta) + L_2^{-1}(u w_x). \end{aligned} \tag{3.18}$$

The complete solution set of system as follows

$$\begin{aligned} u_0(x,t) &= e^{\beta x + \alpha t} + L_1^{-1}(\beta), \\ u_{k+1}(x,t) &= -L_1^{-1} A_k, \quad k \geq 0, \\ w_0(x,t) &= e^{-\beta x - \alpha t} + L_2^{-1}(\beta), \\ w_{k+1}(x,t) &= L_2^{-1} B_k, \quad k \geq 0. \end{aligned} \tag{3.19}$$

Therefore

$$\begin{aligned} u_0(x,t) = e^{\beta x + \alpha t} + L_1^{-1}(\beta) &= e^{\beta x + \alpha t} + e^{\alpha dt} \int_0^t e^{-\alpha dt}(\beta) dt \\ &= e^{\beta x + \alpha t} + \frac{e^{\alpha t}\beta}{\alpha} - \frac{\beta}{\alpha}, \end{aligned}$$

$$\begin{aligned} w_0(x,t) = e^{-\beta x - \alpha t} + L_2^{-1}(\beta) &= e^{-\beta x - \alpha t} + e^{-\alpha dt} \int_0^t e^{\alpha dt}(\beta) dt \\ &= e^{-\beta x - \alpha t} - \frac{e^{-\alpha t}\beta}{\alpha} + \frac{\beta}{\alpha}. \end{aligned}$$

$$\begin{aligned} u_1(x,t) = -L_1^{-1} A_0 &= -e^{\alpha t} \int_0^t e^{-\alpha t}(w_0 u_{0x}) dt \\ &= -\frac{\beta^2}{\alpha} t e^{\beta x + \alpha t} + \frac{\beta^2}{\alpha^2} e^{\beta x + \alpha t} - \frac{\beta}{\alpha} e^{\alpha t} - \frac{\beta^2}{\alpha^2} e^{\beta x} + \frac{\beta}{\alpha}, \end{aligned}$$

$$\begin{aligned} w_1(x,t) = L_2^{-1} B_0 &= e^{-\alpha t} \int_0^t e^{\alpha t}(u_0 w_{0x}) dt \\ &= \frac{\beta^2}{\alpha} t e^{-\beta x - \alpha t} + \frac{\beta^2}{\alpha^2} e^{-\beta x - \alpha t} + \frac{\beta}{\alpha} e^{-\alpha t} - \frac{\beta^2}{\alpha^2} e^{-\beta x} - \frac{\beta}{\alpha}. \end{aligned}$$

$$\vdots$$

The decomposition series solution for the system which we obtained by MADM are given by:

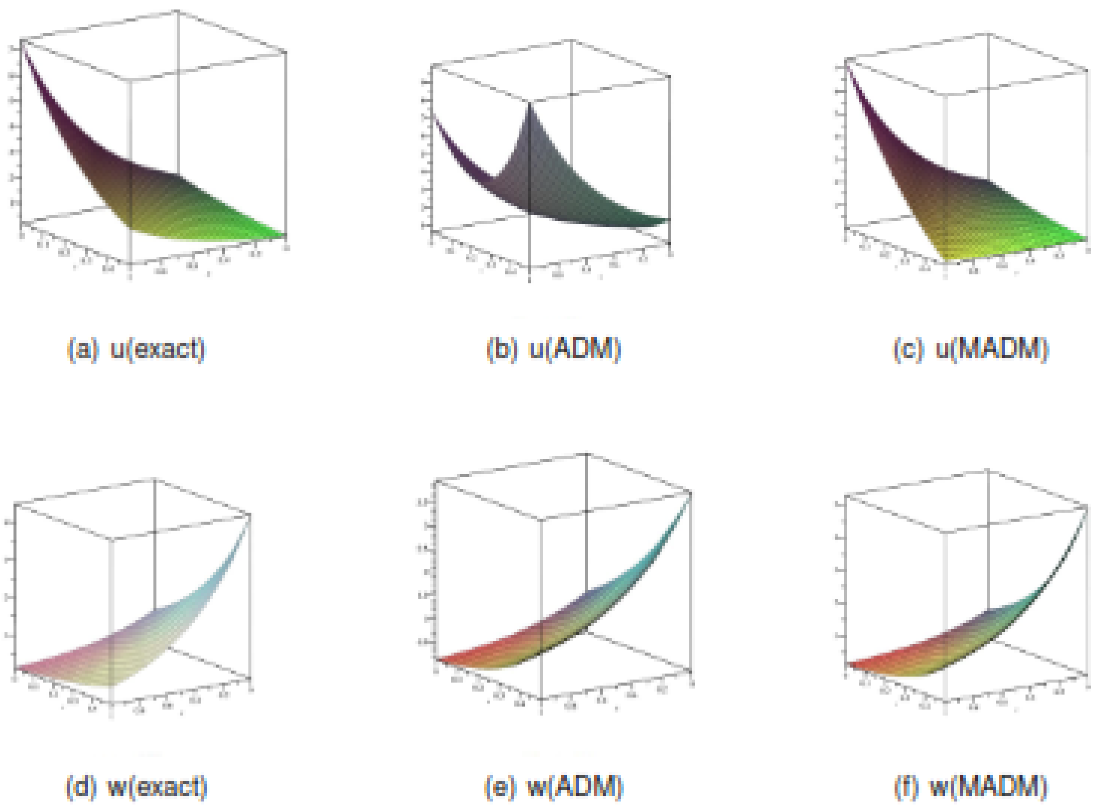
$$\begin{aligned} u(x,t) &= u_0 + u_1 + \cdots \\ &= e^{\beta x + \alpha t} - \frac{\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{\beta^2}{\alpha} t e^{\beta x + \alpha t} + \frac{\beta^2}{\alpha} t e^{\beta x + \alpha t} + \frac{\beta^4}{2\alpha^2} t^2 e^{\beta x + \alpha t} + \cdots \\ &= e^{\beta x + \alpha t} + \frac{\beta^4}{2\alpha^2} t^2 e^{\beta x + \alpha t} + \cdots \end{aligned}$$

$$\begin{aligned} w(x,t) &= w_0 + w_1 + \cdots \\ &= e^{-\beta x-\alpha t} + \frac{\beta}{\alpha} - \frac{\beta}{\alpha} + \frac{\beta^2}{\alpha} t e^{-\beta x-\alpha t} - \frac{\beta^2}{\alpha} t e^{-\beta x-\alpha t} + \frac{\beta^4}{2\alpha^2} t^2 e^{-\beta x-\alpha t} + \cdots \\ &= e^{-\beta x-\alpha t} + \frac{\beta^4}{2\alpha^2} t^2 e^{-\beta x-\alpha t} + \cdots \end{aligned}$$

The decomposition series solution for system (3.15)which we got via ADM are given by:

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + \cdots \\ &= e^{\beta x} + \alpha t e^{\beta x} + \alpha^2 \frac{t^2}{2} e^{\beta x} + \alpha^2 \beta \frac{t^3}{6} - \alpha\beta^2 \frac{t^3}{3} e^{\beta x} + \beta^3 \frac{t^3}{3} + \beta^4 \frac{t^4}{8} e^{\beta x} + \cdots \end{aligned}$$

$$\begin{aligned} w(x,t) &= w_0 + w_1 + w_2 + \cdots \\ &= e^{-\beta x} - \alpha t e^{-\beta x} + \alpha^2 \frac{t^2}{2} e^{-\beta x} + \alpha^2 \beta \frac{t^3}{6} + \alpha\beta^2 \frac{t^3}{3} e^{-\beta x} + \beta^3 \frac{t^3}{3} + \beta^4 \frac{t^4}{8} e^{-\beta x} + \cdots \end{aligned}$$



(a) u(exact) (b) u(ADM) (c) u(MADM)

(d) w(exact) (e) w(ADM) (f) w(MADM)

Fig. 2. 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM at α =-3, β =2 for Example(3.3)

Example 3.4. *Consider the system of nonlinear PDEs*

$$\begin{aligned} u_t + vu_x - 2u = 1, \\ v_t - uv_x + 2v = 1. \end{aligned} \tag{3.20}$$

With initial conditions

$$u(x, 0) = e^x,$$
$$v(x, 0) = e^{-x}.$$

This example is a special case of example (3.2), with $\alpha = 2, \beta = 1$ *and the exact solution is* $u = e^{x+t}, v = e^{-x-t}$.

The system can be expressed as follows

$$\begin{aligned} L_1 u = 1 - vu_x, \\ L_2 v = 1 + uv_x, \end{aligned} \tag{3.21}$$

where $L_1 = e^{\int 2dt} \frac{d}{dt} e^{-\int 2dt}$, $L_2 = e^{-\int 2dt} \frac{d}{dt} e^{\int 2dt}$,

$L_1^{-1}() = e^{\int 2dt} \int_0^t e^{-\int 2dt}()\ dt$, $L_2^{-1}() = e^{-\int 2dt} \int_0^t e^{\int 2\ dt}()dt$.

Using L_1^{-1}, L_2^{-1} *for* $u_t - 2u$, $v_t + 2v$ *of the system (3.20) respectively, we may obtain*

$$\begin{aligned} L_1^{-1}(u_t - 2u) = e^{2t} \int_0^t e^{-2t}(u_t - 2u)\ dt = u - e^{x+2t}, \\ L_2^{-1}(v_t + 2v) = e^{-2t} \int_0^t e^{2t}(v_t + 2v)\ dt = v - e^{-x-2t}. \end{aligned} \tag{3.22}$$

Applying L_1^{-1}, L_2^{-1} *for equations of the system (3.21) respectively, we may obtain*

$$\begin{aligned} u(x, t) = e^{x+2t} + L_1^{-1}(1) - L_1^{-1} vu_x, \\ v(x, t) = e^{-x-2t} + L_2^{-1}(1) + L_2^{-1} uv_x. \end{aligned} \tag{3.23}$$

The general solution of the system is as follows

$$\begin{aligned} u_0(x, t) &= e^{x+2t} + L_1^{-1}(1), \\ u_{k+1}(x, t) &= -L_1^{-1} A_k, \quad k \geq 0, \\ v_0(x, t) &= e^{-x-2t} + L_2^{-1}(1), \\ v_{k+1}(x, t) &= +L_2^{-1} B_k, \quad k \geq 0. \end{aligned} \tag{3.24}$$

The first two solutions iterations are as follows

$$\begin{aligned} u_0 &= e^{x+2t} + \frac{e^{2t}}{2} - \frac{1}{2}, \\ u_1 &= -L_t^{-1} A_0 = -\frac{1}{2} te^{x+2t} + \frac{1}{4} e^{x+2t} - \frac{1}{2} e^{2t} - \frac{1}{4} e^x + \frac{1}{2}, \\ &\vdots \\ v_0 &= e^{-x-2t} - \frac{e^{-2t}}{2} + \frac{1}{2}, \\ v_1 &= L_t^{-1} B_0 = \frac{1}{2} te^{-x-2t} + \frac{1}{4} e^{-x-2t} + \frac{1}{2} e^{-2t} - \frac{1}{4} e^{-x} - \frac{1}{2}, \\ &\vdots \end{aligned}$$

Summing the above iterations yields

$$\begin{aligned} u_{app}(x,t) &= u_0 + u_1 + \cdots = \frac{5}{4}e^{x+2t} + \frac{1}{2}te^{x+2t} - \frac{1}{4}e^{x} + \cdots \\ v_{app}(x,t) &= v_0 + v_1 + \cdots = \frac{5}{4}e^{-x-2t} + \frac{1}{2}te^{-x-2t} - \frac{1}{4}e^{-x} + \cdots \end{aligned} \tag{3.25}$$

Which are the series of the approximate solution for $u(x,t)$and $v(x,t)$ given using MADM. And

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots = e^{x} + 2te^{x} + 2t^2e^{x} + t^3 - 2\frac{t^3}{3}e^{x} + \frac{t^4}{8}e^{x} + \cdots \\ v &= \sum_{n=0}^{\infty} v_n = v_0 + v_1 + v_2 + \cdots = e^{-x} - 2te^{-x} + 2t^2e^{-x} + t^3 + 2\frac{t^3}{3}e^{-x} + \frac{t^4}{8}e^{-x} + \cdots \end{aligned} \tag{3.26}$$

Which are the series of the approximate solution for u(x,t), v(x,t) that we produced via ADM.

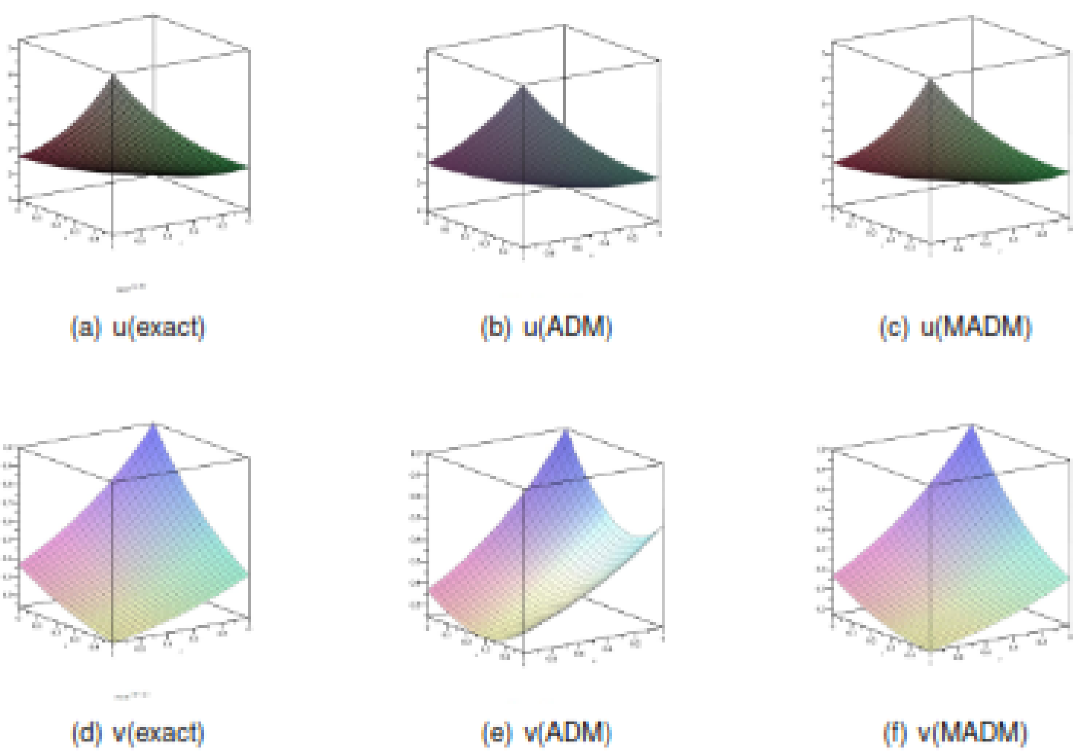


Fig. 3. 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM for Example(3.4)

Example 3.5. *Consider the system of nonlinear PDEs*

$$\begin{aligned} \frac{\partial u}{\partial t} + \left(v\frac{\partial u}{\partial x}\right)^2 + u = 1, \\ \frac{\partial v}{\partial t} + \left(u\frac{\partial v}{\partial x}\right)^2 - v = 1. \end{aligned} \tag{3.27}$$

With initial conditions

$$u(x,0) = e^x,$$
$$v(x,0) = e^{-x}.$$

The exact solution is given in as (Biazar et al., 2007; Alderremy et al., 2022)

$$u(x,t) = e^{x-t},$$
$$v(x,t) = e^{-x+t}.$$

For the modified method, the system is expressed as follows

$$\begin{aligned} L_1 u = 1 - (vu_x)^2, \\ L_2 v = 1 - (uv_x)^2, \end{aligned} \tag{3.28}$$

where $L_1 = e^{-\int\ dt}\frac{d}{dt}e^{\int\ dt}$, $L_2 = e^{\int\ dt}\frac{d}{dt}e^{-\int\ dt}$,
$L_1^{-1}() = e^{-\int\ dt}\int_0^t e^{\int\ dt}()\ dt$, $L_2^{-1}() = e^{\int\ dt}\int_0^t e^{-\int\ dt}()\ dt$.
Using L_1^{-1}, L_2^{-1} *for* $u_t + u$, $v_t - v$, *of the system (3,27) respectively, it produces the following*

$$\begin{aligned} L_1^{-1}(u_t + u) = e^{-t}\int_0^t e^t(u_t + u)\ dt = u - e^{x-t}, \\ L_2^{-1}(v_t - v) = e^t\int_0^t e^{-t}(v_t - v)\ dt = u - e^{-x+t}. \end{aligned} \tag{3.29}$$

Employing L_1^{-1}, L_2^{-1} *for equations of the system (3.28) respectively, we may obtain*

$$\begin{aligned} u(x,t) = e^{x-t} + L_1^{-1}(1) - L_1^{-1}(vu_x)^2 \\ v(x,t) = e^{-x+t} + L_2^{-1}(1) - L_2^{-1}(uv_x)^2 \end{aligned} \tag{3.30}$$

The general solution of system as follows

$$\begin{aligned} u_0(x,t) &= e^{x-t} + L_1^{-1}(1), \\ u_{k+1}(x,t) &= -L_1^{-1}A_k^2, \quad k \geq 0, \\ v_0 &= e^{-x+t} + L_2^{-1}(1), \\ v_{k+1}(x,t) &= -L_2^{-1}B_k^2, \quad k \geq 0. \end{aligned} \tag{3.31}$$

The first two terms of solutions iterations are as follows

$$\begin{aligned} u_0(x,t) &= e^{x-t} + 1 - e^{-t} \\ v_0(x,t) &= e^{-x+t} - 1 + e^t \\ u_1(x,t) &= -L_1^{-1}A_0^2 = -e^{-t}\int_0^t e^t(v_0 u_{0_x})^2\ dt \\ &= 2te^{2x-t} + 2te^{x-t} + e^{2x-2t} + 2e^{x-t} + e^{-t} - e^{2x} - 2e^x - 1 \\ v_1(x,t) &= -L_2^{-1}B_0^2 = -e^t\int_0^t e^{-t}(u_0 v_{0_x})^2\ dt \\ &= 2te^{-2x+t} - 2te^{-x+t} - e^{-2x+2t} + 2e^{-x+t} - e^t + e^{-2x} - 2e^{-x} + 1 \end{aligned}$$

$$\vdots$$

Summing the above iterations yields

$$\begin{aligned} u(x,t) &= u_0+u_1+\cdots = 2te^{2x-t}+2te^{x-t}+e^{2x-2t}+3e^{x-t}-e^{2x}-2e^{x}+\cdots \\ v(x,t) &= v_0+v_1+\cdots = 2te^{-2x+t}-2te^{-x+t}-e^{-2x+2t}+2e^{-x+t}+e^{-2x}-2e^{-x}+\cdots \end{aligned}$$

Which the approximate solutions for u(x,t), v(x,t) of the system that we obtained by MADM are given.

And the approximate solutions for u(x,t), v(x,t) of the system that we obtained by ADM are given.

$$\begin{aligned} u(x,t)=u_0+u_1+u_2+\cdots &= e^{x}+t-\frac{e^{2x}t^3}{3}-e^{x}t^2-\frac{t^2}{2}-t-te^{x} \\ &- \frac{4}{81}t^9e^{4x}-\frac{1}{18}(e^{x}+e^{3x})t^8 \\ &- \frac{1}{7}\left[\frac{4}{3}(2e^{2x}+\frac{1}{2}e^{3x})-\frac{1}{9}(e^{-x}+5e^{x})^2\right]t^7 \\ &- \frac{1}{6}\left[\frac{4}{3}e^{3x}+\frac{2}{3}(2+\frac{1}{2}e^{x})(e^{-x}+e^{5x})\right]t^6 \\ &- \frac{1}{5}\left[\frac{2}{3}(1+5e^{2x})-(2+\frac{1}{2}e^{x})^2\right]t^5 \\ &- \frac{1}{4}\left[2e^{x}(2+\frac{1}{2}e^{x})-\frac{1}{3}e^{2x}\right]t^4 \\ &- \frac{1}{3}\left[e^{2x}-e^{x}-\frac{1}{2}\right]t^3+\frac{1}{2}\left[1+e^{x}\right]t^2+\cdots \end{aligned}$$

$$\begin{aligned} v(x,t)=v_0+v_1+v_2+\cdots &= e^{-x}+t-\frac{e^{-2x}t^3}{3}-e^{-x}t^2+\frac{t^2}{2}-t+te^{-x} \\ &- \frac{4}{81}t^9e^{-4x}-\frac{1}{18}(e^{-x}+e^{-3x})t^8 \\ &- \frac{1}{7}\left[\frac{4}{3}(2e^{-2x}-\frac{1}{2}e^{-3x})+\frac{1}{9}(e^{x}+5e^{-x})^2\right]t^7 \\ &- \frac{1}{6}\left[\frac{4}{3}e^{-3x}+\frac{2}{3}(2-\frac{1}{2}e^{-x})(e^{x}+e^{-5x})\right]t^6 \\ &- \frac{1}{5}\left[\frac{2}{3}(1+5e^{-2x})+(2-\frac{1}{2}e^{-x})^2\right]t^5 \\ &- \frac{1}{4}\left[2e^{-x}(2-\frac{1}{2}e^{-x})+\frac{1}{3}e^{-2x}\right]t^4 \\ &- \frac{1}{3}\left[e^{-2x}+e^{-x}-\frac{1}{2}\right]t^3-\frac{1}{2}\left[1-e^{x}\right]t^2+\cdots \end{aligned}$$

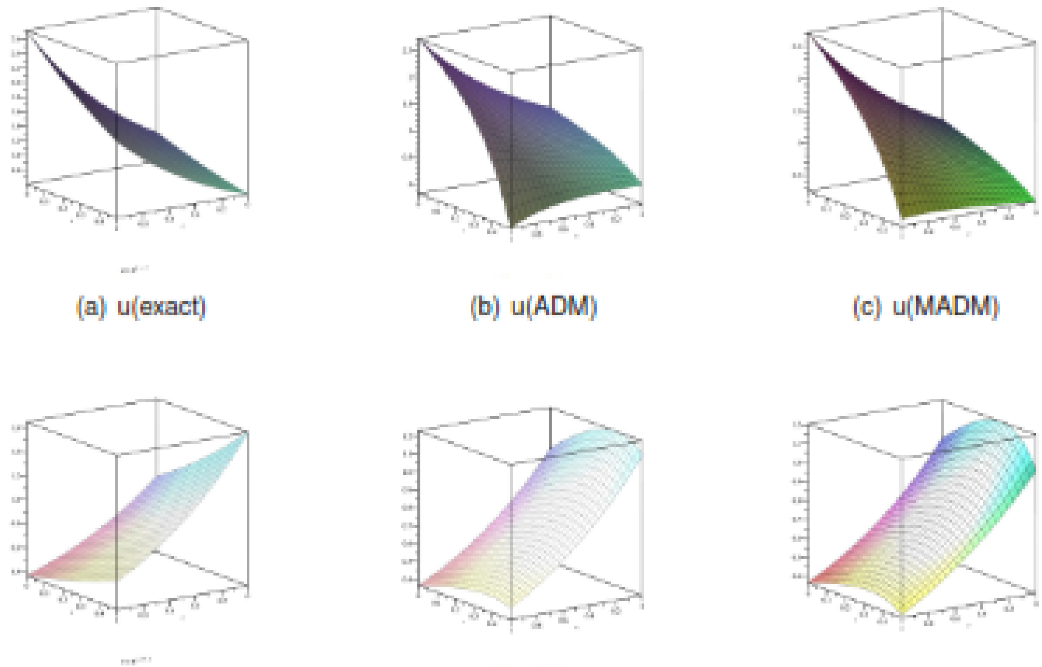
Remark: By using the noise terms phenomenon in the above example

$$\begin{aligned} u_0(x,t) &= u_0\Phi(0)\Phi^{-1}(t)=e^{x-t}, \quad \Phi^{-1}(t)=e^{-\int dt}, \\ v_0(x,t) &= v_0\Phi(0)\Phi^{-1}(t)=e^{-x+t}, \quad \Phi^{-1}(t)=e^{\int dt}, \\ u_1(x,t) &= L_1^{-1}(1)-L_1^{-1}A_0^2=0, \\ v_1(x,t) &= L_2^{-1}(1)-L_2^{-1}B_0^2=0, \end{aligned}$$

$$\vdots$$

$$u_{k+1}(x,t) = -L_1^{-1}A_k^2 = 0, \quad k \geq 1,$$
$$v_{k+1}(x,t) = -L_2^{-1}B_k^2 = 0, \quad k \geq 1.$$

We have
$(u, v) = (u_0(x,t), v_0(x,t)) = (e^{x-t}, e^{-x+t})$,
which represents a closed solution. This also applies to Ex(3.3)and Ex(3.4)



(a) u(exact) (b) u(ADM) (c) u(MADM)

(d) v(exact) (e) v(ADM) (f) v(MADM)

Fig. 4. 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM for Example(3.5)

4 Conclusion

This research demonstrates that the Modified Adomian Decomposition Method (MADM) is a valuable tool for tackling complex nonlinear partial differential equations. While both MADM and the standard Adomian Decomposition Method (ADM) are effective for linear systems, MADM offers improved convergence and accuracy due to the incorporation of a new integral operator. The successful application of MADM in the presented numerical examples expands the capabilities of Adomian methods for solving a wider range of nonlinear PDEs.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Authors have declared that no competing interests exist.

References

Adomian, G., & Rach, R. (1992). Noise terms in decomposition series solution. Computers & Mathematics with Applications, 24(11), 61.

Adomian, G. (1990). A review of the decomposition method and some recent results for nonlinear equations. Mathematics and Computers in Simulation, 13(7), 17-43.

Aland, P. V., & Singh, P. (2022). Solution of nonlinear partial differential equations using Laplace transform modified Adomian decomposition method. Journal of Physics: Conference Series, 28(1), 012156.

Alomari, S. N., & Hasan, Y. Q. (2023). Solving the hyperbolic telegraph equation using a modified Adomian decomposition method with an invertible partial differential operator. Asian Research Journal of Mathematics, 19(9), 278-293.

Alsidrani, F., Kılıçman, A., & Senu, N. (2024). A comprehensive review of the recent numerical methods for solving fractional partial differential equations. Open Mathematics, 22(1), 20240036.

Alderremy, A. A., Shah, R., Iqbal, N., Aly, S., & Nonlaopon, K. (2022). Fractional series solution construction for nonlinear fractional reaction-diffusion Brusselator model utilizing Laplace residual power series. Symmetry, 14, 1944.

Almardy, A., Belkhamsa, M., Farah, R. A., Saadouli, H., Alkeer, M. A., Mohammed, M. A., & Osman, A. K. (2023). Analytical solutions of some special nonlinear partial differential equations using Aboodh-Adomian decomposition method. Journal of Positive School Psychology, 7, 474-481.

AL-Mazmumy, M., Alsulami, A. A., Bakodah, H. O., & Alzaid, N. (2022). Modified Adomian method through efficient inverse integral operators to solve nonlinear initial-value problems for ordinary differential equations. Axioms, 11(12), 698.

Babolian, E., Biazar, J., & Vahidi, A. R. (2004). Solution of a system of nonlinear equations by Adomian decomposition method. Journal of Mathematical Analysis and Applications, 50(3), 847-854.

Biazar, J., Eslami, M., & Ghazvini, H. (2007). Homotopy perturbation method for systems of partial differential equations. International Journal of Nonlinear Science and Numerical Simulation, 8(3), 413-418.

Daftardar-Gejji, V., & Jafari, H. (2005). Adomian decomposition: A tool for solving a system of fractional differential equations. Journal of Mathematical Analysis and Applications, 301(2), 508-518.

Dubey, V. P., Kumar, R., & Kumar, D. (2020). An efficient computational scheme for nonlinear time fractional systems of partial differential equations arising in physical sciences. Advances in Difference Equations, 2020, 1-27.

Ebiwareme, L. (2022). A comparative study between semi-analytical iterative schemes for the reliable treatment of systems of coupled nonlinear partial differential equations. International Journal of Innovation Engineering and Science Research, 6(1).

Fadaei, J. (2011). Application of Laplace-Adomian decomposition method on linear and nonlinear system of PDEs. Applied Mathematical Sciences, 5(27), 1307-1315.

Hasan, Y. Q. (2014). A new development to the Adomian decomposition for solving singular IVPs of Lxne-Emden type. United States of America Research Journal (USARJ), 2(3), 9-13.

Hasan, Y. Q. (2012). Solving first-order ordinary differential equations by modified Adomian decomposition method. Advances in Intelligent Transportation Systems, 1(4), 86-89.

Kanth, A. R., & Aruna, K. (2008). Differential transform method for solving linear and nonlinear systems of partial differential equations. Physics Letters A, 372(46), 6896-6898.

Khan, H., Shah, R., Kumam, P., Baleanu, D., & Arif, M. (2020). Laplace decomposition for solving nonlinear systems of fractional order partial differential equations. Advances in Difference Equations, 2020, 1-18.

Kaya, D., & El-Sayed, S. M. (2004). A numerical solution of the Klein–Gordon equation and convergence of the decomposition method. Applied Mathematics and Computation, 156(2), 341-353.

Loyinmi, A. C. (2024). Analysis of two hybrid schemes to solve the Benjamin-Bona-Mahony (BBM) equation. Kathmandu University Journal of Science, Engineering and Technology, 18(1), 1-15.

Loufouilou, J. M., Yindoula, J. B., & Bissanga, G. (2021). Application of the Adomian decomposition method (ADM) to solving systems of partial differential equations. International Journal of Applied Mathematics and Theoretical Physics, 7(1), 28.

Nuruddeen, R. I., Muhammad, L., Nass, A. M., & Sulaiman, T. A. (2018). A review of integral transform-based decomposition methods and their applications in solving nonlinear PDEs. Palestine Journal of Mathematics, 7(1), 262-280.

Rab, A., Khan, S., Khan, H., Tchier, F., Jebran, S., Tawfiq, F., & Nadeem, M. (2024). The fractional analysis of thermo-elasticity coupled systems with non-linear and singular nature. Scientific Reports, 14(1), 9663.

Sekgothe, N. H. (2021). Application of Adomian decomposition method to solving nonlinear differential equations (Doctoral dissertation).

Wazwaz, A. M. (2007). The variational iteration method for solving linear and nonlinear systems of PDEs. Computers & Mathematics with Applications, 54(7-8), 895-902.

Wazwaz, A. M. (2000). The decomposition method applied to a system of partial differential equations and to the reaction-diffusion Brusselator model. Applied Mathematics and Computation, 110(2-3), 251-264.

Yavuz, M. (2019). Dynamical behaviors of separated homotopy method defined by conformable operator. Konuralp Journal of Mathematics, 7(1), 1-6.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/124434