



Odd Lindley-Kumaraswamy Distribution: Model, Properties and Application

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript

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Abstract

This article uses the odd Lindley-G family of distributions to propose and study a new compound distribution called “odd Lindley-Kumaraswamy distribution”. In this article, the density and distribution functions of the odd Lindley-Kumaraswamy distribution are defined and studied by deriving and discussing many properties of the distribution such as the ordinary moments, moment generating function, characteristics function, quantile function, reliability functions, order statistics and other useful measures. The unknown model parameters are also estimated by the method of maximum likelihood. The goodness-of-fit of the proposed distribution is demonstrated using two real life datasets. The results show that the proposed distribution outperforms the other fitted compound models selected for this study and hence it is a flexible generalization of the Kumaraswamy distribution.

Keywords: Odd Lindley-G family; kumaraswamy distribution; odd lindley-kumaraswamy distribution; statistical properties; parameter estimation; goodness-of-fit tests and flexibility.

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1 Introduction

The Kumaraswamy distribution describes a distribution in which outcomes are limited to a specific range, the probability density function within this range being characterized by two shape parameters. It is similar to the beta distribution but possibly easier to use because it has simpler analytical expressions for its probability density function and cumulative distribution function. The Kumaraswamy probability distribution was originally proposed by [1] for double bounded random processes for hydrological applications. The Kumaraswamy distribution is a family of continuous probability distributions defined on the interval $[0,1]$ with cumulative distribution function (*cdf*) given by

$$G(x) = 1 - (1 - x^\alpha)^\beta \quad (1)$$

And the corresponding probability density function (*pdf*) given by

$$g(x) = \alpha\beta x^{\alpha-1} (1 - x^\alpha)^{\beta-1} \quad (2)$$

For $0 \leq x \leq 1$, where $\alpha > 0$ and $\beta > 0$ are the shape parameters.

There are several methods of extending standard distributions and among the very recent ones are Logistic-X family [2], a new Weibull-G family [3], a Lindley-G family [4], a Gompertz-G family [5], an odd Lindley-G family [6] and an odd Lomax generator of distributions (Odd Lomax-G family) [7].

Due to this families and many others, numerous compound models have been introduced for modeling statistical data and these compound distributions are found to be skewed, flexible and more better in statistical modeling than the standard counterparts ([8-23]).

In order to add skewness level and flexibility to the conventional Kumaraswamy distribution, these families of distribution have also led to the development of some generalizations of the Kumaraswamy distribution in the literature such as the transmuted Kumaraswamy distribution [24], the exponentiated Kumaraswamy distribution [25], the Kumaraswamy- Kumaraswamy distribution [26] and the Lomax-Kumaraswamy distribution [27].

According to [6], the odd Lindley-Weibull distribution which is based on the odd Lindley-G family was found to fit real dataset better compared exponentiated Weibull distribution, beta Weibull distribution, Kumaraswamy Weibull distribution and the conventional Weibull distribution. Also, according to [22], the odd Lindley-Rayleigh distribution which is also based on the odd Lindley-G family performed better than the transmuted Weibull-Rayleigh distribution, Weibull-Rayleigh distribution, transmuted Rayleigh distribution, Lindley distribution and the Rayleigh distribution after real life data applications.

Hence, our interest in this article is to also develop a new extension of the Kumaraswamy distribution with greater flexibility based on the odd Lindley-G family of probability distributions proposed by [6].

The following is the arrangement of remaining sections in this article: the new model with graphical representation is given in section 2. Section 3 derived some properties of the new distribution. The distribution of order statistics is presented in section 4. The estimation of unknown parameters of the distribution using maximum likelihood estimation is provided in section 5. An application of the new model to a real life dataset is done in section 6 and a summary and conclusion of the study is given in section 7.

2 Odd Lomax-Kumaraswamy Distribution (OLKD)

According to [6], the cumulative distribution function (cdf) and the probability density function (pdf) of the Odd Lindley-G family of distributions are defined as:

$$F(x) = \int_{-\infty}^{\frac{G(x)}{1-G(x)}} \frac{\theta^2}{\theta+1} (1+t) e^{-\theta t} dt = 1 - \frac{\theta + (1-G(x))}{(1+\theta)(1-G(x))} \exp \left\{ -\theta \left[\frac{G(x)}{1-G(x)} \right] \right\} \quad (3)$$

And

$$f(x) = \frac{\theta^2 g(x)}{(1+\theta)(1-G(x))^3} \exp \left\{ -\theta \left[\frac{G(x)}{1-G(x)} \right] \right\} \quad (4)$$

respectively, where $g(x)$ and $G(x)$ are the *pdf* and the *cdf* of any continuous distribution to be modified respectively and $\theta > 0$ is the shape parameter of the family responsible for additional skewness and flexibility in the modified model.

Substituting equation (1) and (2) in (3) and (4) above and simplifying, we obtain the cdf and pdf of the OLinKumD for a random variable X as:

$$F(x) = 1 - \frac{\theta + (1-x^\alpha)^\beta}{(1+\theta)(1-x^\alpha)^\beta} e^{-\theta \left[(1-x^\alpha)^{-\beta} - 1 \right]} \quad (5)$$

And

$$f(x) = \frac{\alpha \beta \theta^2 x^{\alpha-1} (1-x^\alpha)^{\beta-1}}{(1+\theta)(1-x^\alpha)^{3\beta}} e^{-\theta \left[(1-x^\alpha)^{-\beta} - 1 \right]} \quad (6)$$

“respectively, for $\alpha, \beta, \theta > 0$ and $0 \leq x \leq 1$ s, where $\alpha > 0$, $\theta > 0$ and $\beta > 0$ are the shape parameters.

The plot of the pdf and cdf of the OLinKumD using some parameter values are displayed in Figs. 1 and 2 as follows.

Fig. 1 shows that the proposed distribution can take different shapes due to the plots developed under various parameter values and it is hence a flexible model.

3 Properties of Odd Lindley-Kumaraswamy Distribution

This section contains derivations and discussions of some properties of the proposed distribution. These are presented as follows:

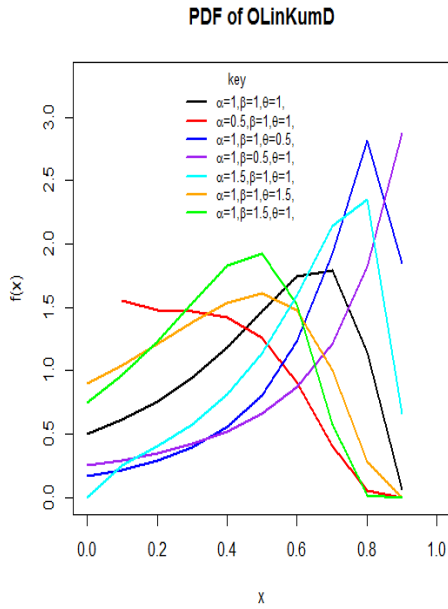


Fig. 1. PDF of OLinKumD.

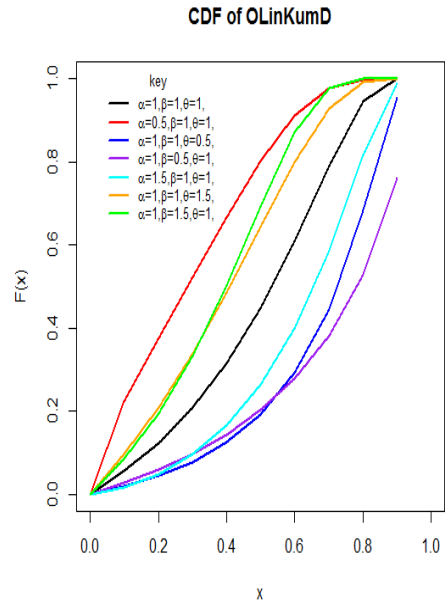


Fig. 2. CDF of OLinKumD.

3.1 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu'_n = E(X^n) = \int_0^{\infty} x^n f(x) dx \tag{7}$$

where $f(x)$, the pdf of the OLinKumD is as given in equation (6) as:

$$f(x) = \frac{\alpha\beta\theta^2 x^{\alpha-1} (1-x^\alpha)^{\beta-1}}{(1+\theta)(1-x^\alpha)^{3\beta}} e^{-\theta[(1-x^\alpha)^{-\beta}-1]}$$

Simplifying the pdf above results in the following:

$$f(x) = \frac{\alpha\beta\theta^2}{(1+\theta)} x^{\alpha-1} (1-x^\alpha)^{-2\beta-1} e^{\theta[1-(1-x^\alpha)^{-\beta}]} \tag{8}$$

Before substituting (8) in (7), we perform the expansion and simplification and linear representation of the pdf as follows:

First, by using power series expansion on the last term in (8), we obtain:

$$e^{\theta[1-(1-x^\alpha)^{-\beta}]} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} [1-(1-x^\alpha)^{-\beta}]^k \tag{9}$$

Making use of the result in (9) above and simplifying, equation (8) becomes

$$f(x) = \frac{\alpha\beta\theta^2}{(1+\theta)} \sum_{k=0}^{\infty} \frac{\theta^k}{k!} x^{\alpha-1} (1-x^\alpha)^{-2\beta-1} [1-(1-x^\alpha)^{-\beta}]^k \tag{10}$$

Also using generalized binomial expansion gives:

$$[1-(1-x^\alpha)^{-\beta}]^k = \sum_{m=0}^{\infty} (-1)^m \binom{k}{m} (1-x^\alpha)^{-\beta m} \tag{11}$$

Making use of the result in (11) above in equation (10) and simplifying, we obtain:

$$f(x) = \frac{\alpha\beta\theta^2}{(1+\theta)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m \theta^k}{k!} \binom{k}{m} x^{\alpha-1} (1-x^\alpha)^{-\beta(m+2)-1} \tag{12}$$

Now, let $\eta_{km} = \frac{\theta^2}{(1+\theta)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m \theta^k}{k!} \binom{k}{m}$ be a constant, which implies that the pdf in (12) can also be written in its simple and linear form as:

$$f(x) = \eta_{km} \alpha\beta x^{\alpha-1} (1-x^\alpha)^{-\beta(m+2)-1} \tag{13}$$

Now, using the linear representation of the pdf of the OLinKumD in equation (13), the n^{th} ordinary moment of the OLinKumD is represented as:

$$\mu'_n = E(X^n) = \int_0^1 x^n f(x) dx = \int_0^1 \eta_{km} \alpha\beta x^{n+\alpha-1} (1-x^\alpha)^{-\beta(m+2)-1} dx \tag{14}$$

Recall the for the Kumaraswamy distribution, the r^{th} ordinary moment is given as:

$$\mu'_r = E(X^r) = \int_0^1 x^r f(x) dx = \alpha\beta \int_0^1 x^{r+\alpha-1} (1-x^\alpha)^{\beta-1} dx = \beta B\left(\frac{r}{\alpha} + 1, \beta\right) \tag{15}$$

Therefore, the n^{th} ordinary moment of the OLKD can be expressed from (15) as:

$$\mu'_n = E(X^n) = \int_0^1 \eta_{km} \alpha\beta x^{n+\alpha-1} (1-x^\alpha)^{-\beta(m+2)-1} dx = \eta_{km} \beta B\left(\frac{n}{\theta} + 1, \beta(m+2)\right)$$

$$\mu'_n = E(X^n) = \frac{\beta\theta^2}{(1+\theta)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m \theta^k}{k!} \binom{k}{m} B\left(\frac{n}{\theta} + 1, \beta(m+2)\right) \quad (16)$$

The mean (μ'_1), variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) can be calculated from the ordinary and non-central moments using some well-known relationships such as:

$$\mu'_1 = E(X), Var(X) = \sigma^2 = \mu'_2 - \{\mu'_1\}^2, CV = \left\{ \frac{\sigma^2}{\{\mu'_1\}^2} \right\}^{\frac{1}{2}} \quad CS = E\left(\frac{x - \mu'_1}{\sigma}\right)^3 = \frac{\mu_3}{(\sigma)^3} \quad \text{and}$$

$$CK = E\left(\frac{x - \mu'_1}{\sigma}\right)^4 = \frac{\mu_4}{(\sigma)^4}.$$

3.2 Moment generating function

The moment generating function of a random variable X can be obtained as

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx \quad (17)$$

Recall that by power series expansion,

$$e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} = \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r \quad (18)$$

Therefore, the moment generating function can also be expressed as:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^1 x^r f(x) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{r=0}^{\infty} \frac{t^r}{r!} [\mu'_r]$$

Using the result in equation (18) and simplifying the integral in (17) therefore we have:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\frac{\beta\theta^2}{(1+\theta)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m \theta^k}{k!} \binom{k}{m} B\left(\frac{r}{\theta} + 1, \beta(m+2)\right) \right] \quad (19)$$

3.3 Characteristics function

A representation for the characteristics function is given by

$$\phi_x(t) = E(e^{itx}) = \int_0^1 e^{itx} f(x) dx \quad (20)$$

Recall that by power series expansion,

$$e^{itx} = \sum_{r=0}^{\infty} \frac{(itx)^r}{r!} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} x^r \tag{21}$$

Hence, simple algebra and use of (21) above produces the following results:

$$\begin{aligned} \phi_x(t) &= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^1 x^r f(x) dx = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} [\mu_r'] \\ \phi_x(t) &= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\frac{\beta\theta^2}{(1+\theta)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m \theta^k}{k!} \binom{k}{m} B\left(\frac{r}{\theta} + 1, \beta(m+2)\right) \right] \end{aligned} \tag{22}$$

3.4 Quantile function, median and simulation

According to [28], the quantile function for any distribution is defined in the form $Q(u) = F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0 < u < 1$

To derive the quantile function of the OLinKumD, the cdf of the OLinKumD is considered and inverted according to the above definition as follows:

$$F(x) = 1 - \frac{\theta + (1-x^\alpha)^\beta}{(1+\theta)(1-x^\alpha)^\beta} e^{-\theta[(1-x^\alpha)^{-\beta}-1]} = u \tag{23}$$

Simplifying equation (23) above gives:

$$-(1+\theta)(1-u)e^{-(\theta+1)} = -\frac{\theta + (1-x^\alpha)^\beta}{(1-x^\alpha)^\beta} e^{-\frac{\theta + (1-x^\alpha)^\beta}{(1-x^\alpha)^\beta}} \tag{24}$$

In the expression above, it can be seen that $-\frac{\theta + (1-x^\alpha)^\beta}{(1-x^\alpha)^\beta}$ is the Lambert function of the real argument,

$-(1+\theta)(1-u)e^{-(\theta+1)}$ since the Lambert function is defined as: $w(x)e^{w(x)} = x$

Also note that the Lambert function has two branches with a branching point located at $(-e^{-1}, 1)$. The lower branch, $W_{-1}(x)$ is defined in the interval $[-e^{-1}, 1]$ and has a negative singularity for $x \rightarrow 0^{-1}$. The upper branch, $W_0(x)$, is defined for $x \in [-e^{-1}, \infty]$. Hence, equation (24) can be written as:

$$W\left(- (1+\theta)(1-u)e^{-(\theta+1)}\right) = -\frac{\theta + (1-x^\alpha)^\beta}{(1-x^\alpha)^\beta} \tag{25}$$

Now for any $\theta > 0$ and $u \in (0,1)$, it follows that $\frac{\alpha + 1 - e^{-\frac{\theta}{x}}}{1 - e^{-\frac{\theta}{x}}} > 1$ and $\left((1+\theta)(1-u)e^{-(\theta+1)}\right) < 0$.

Therefore, considering the lower branch of the Lambert function, equation (25) can be presented as:

$$W_{-1}\left(-(\alpha+1)(1-u)e^{-(\alpha+1)}\right) = -\frac{\alpha + 1 - e^{-\frac{\theta}{x}}}{1 - e^{-\frac{\theta}{x}}} \tag{26}$$

Collecting like terms in equation (26) and simplifying the result, the quantile function of the OLinKumD is obtained as:

$$Q(u) = \left\{ 1 - \left[-\frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(- (1+\theta)(1-u)e^{-(\theta+1)}\right) \right]^{-\frac{1}{\beta}} \right\}^{\frac{1}{\alpha}} \tag{27}$$

where u is a uniform variate on the unit interval $(0,1)$ and $W_{-1}(\cdot)$ represents the negative branch of the Lambert function.

The median of X from the OLinKumD is simply obtained by setting $u=0.5$ and this substitution of $u = 0.5$ in Equation (27) leads to:

$$MD = \left\{ 1 - \left[-\frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(-\frac{1}{2}(1+\theta)e^{-(\theta+1)}\right) \right]^{-\frac{1}{\beta}} \right\}^{\frac{1}{\alpha}} \tag{28}$$

In a Similar way, random numbers can be simulated from the OLinKumD by setting $Q(u) = X$ and this process is called simulation using inverse transformation method. This means for any $\alpha, \beta, \theta > 0$ and $u \in (0,1)$:

$$X = \left\{ 1 - \left[-\frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(- (1+\theta)(1-u)e^{-(\theta+1)}\right) \right]^{-\frac{1}{\beta}} \right\}^{\frac{1}{\alpha}} \tag{29}$$

“where u is a uniform variate on the unit interval $(0,1)$ and $W_{-1}(\cdot)$ represents the negative branch of the Lambert function”.

Again from the above quantile function, the quantile based measures of skewness and kurtosis are obtained as follows:

According [29], the Bowley’s measure of skewness based on quartiles is defined as:

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \tag{30}$$

And [30] presented the Moors' kurtosis based on octiles as:

$$KT = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{1}{8}\right)} \tag{31}$$

“where $Q(\cdot)$ is calculated by using the quantile function from equation (27).

3.5 Reliability analysis of the OLinKumD

In this section, the survival (or reliability) function and the hazard function are obtained for the OLinKumD.

The Survival function describes the probability that a unit, component or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{32}$$

Using the *cdf* of the OLinKumD in (32) and simplifying the result, the survival function for the OLinKumD is obtained as:

$$S(x) = \frac{\theta + (1 - x^\alpha)^\beta}{(1 + \theta)(1 - x^\alpha)^\beta} e^{-\theta[(1 - x^\alpha)^{-\beta} - 1]} \tag{33}$$

The figure below presents some plots for the survival function of the OLinKumD using different parameter values.

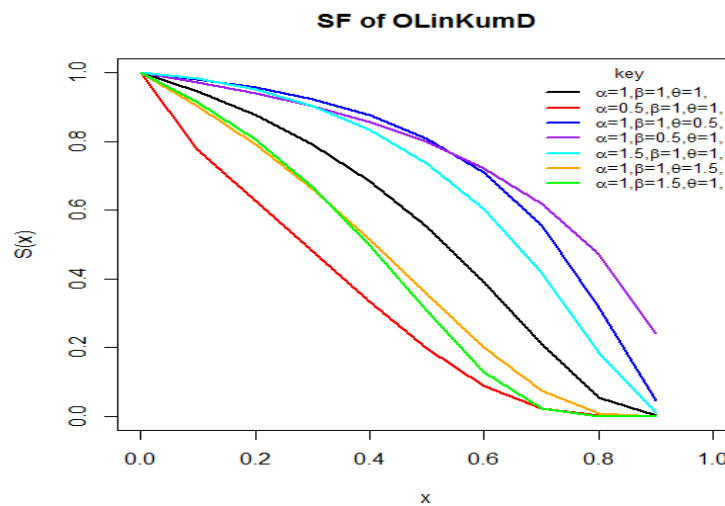


Fig. 3. Survival Function of OLinKumD

Fig. 3. is a plot of the survival function which indicates that the probability of survival for any OLinKumD variable equals one (1) at initial time or early age and it decreases as time increases and equals zero (0) as it approaches infinity.

Hazard function is also called failure rate function and it represents the likelihood that a component will fail for an interval of time. The hazard function is defined as:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1-F(x)} \tag{34}$$

Making use of the pdf and cdf of OLinKumD, an expression for the hazard rate of the OLinKumD is simplified and given by:

$$h(x) = \frac{\alpha\beta\theta^2 x^{\alpha-1}}{\left(\theta + (1-x^\alpha)^\beta\right)(1-x^\alpha)^{\beta+1}} \tag{35}$$

where $0 \leq x \leq 1, \alpha, \beta, \theta > 0$.

The following figure displays useful plots of the hazard function of OLinKumD for some arbitrary parameter values.

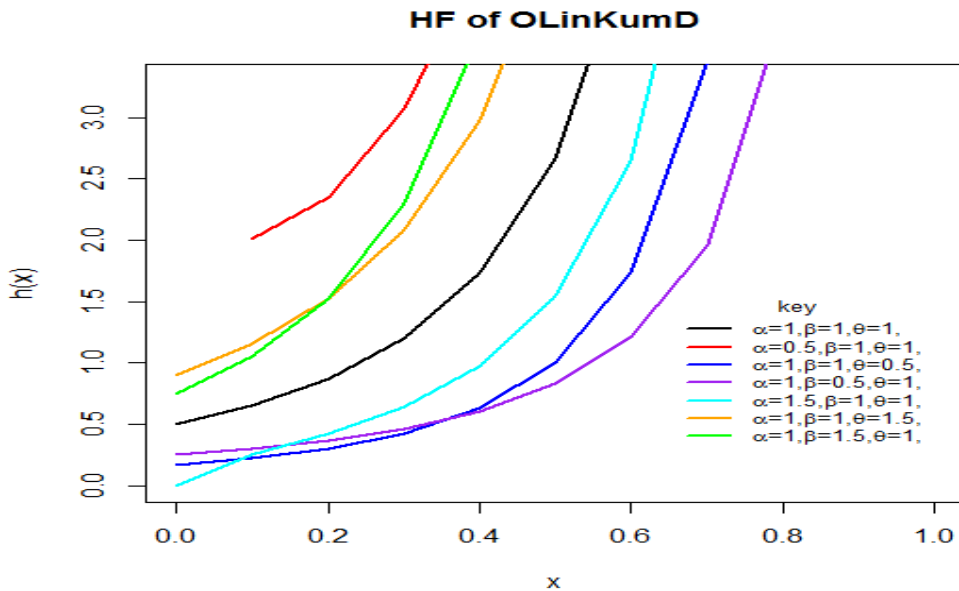


Fig. 4. The Hazard Function of OLinKumD

It can be seen from Fig. 4 that the hazard function increases as X (time) increases. This is evidence that the OLinKumD could be appropriate for modeling time dependent events, where risk or hazard increases with time or age.

4 Order Statistics

Suppose X_1, X_2, \dots, X_n is a random sample from the OLinKumD and let $X_{1:n}, X_{2:n}, \dots, X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The pdf, $f_{a:n}(x)$ of the a^{th} order statistic can be obtained by:

$$f_{a:n}(x) = \frac{n!}{(a-1)!(n-a)!} \sum_{k=0}^{n-a} (-1)^k \binom{n-a}{k} f(x) F(x)^{k+a-1} \quad (36)$$

Using (5) and (6), the pdf of the a^{th} order statistics $X_{a:n}$, can be expressed from (36) as:

$$f_{a:n}(x) = \frac{n!}{(a-1)!(n-a)!} \sum_{k=0}^{n-a} (-1)^k \binom{n-a}{k} \left[\frac{\alpha\beta\theta^2 x^{\alpha-1} e^{-\theta[(1-x^\alpha)^\beta - 1]}}{(1+\theta)(1-x^\alpha)^{2\beta+1}} \right] \left[1 - \frac{\theta + (1-x^\alpha)^\beta}{(1+\theta)(1-x^\alpha)^\beta} e^{-\theta[(1-x^\alpha)^\beta - 1]} \right]^{k+a-1} \quad (37)$$

Hence, the pdf of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the OLinKumD are respectively given by:

$$f_{1:n}(x) = \frac{n!}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[\frac{\alpha\beta\theta^2 x^{\alpha-1} e^{-\theta[(1-x^\alpha)^\beta - 1]}}{(1+\theta)(1-x^\alpha)^{2\beta+1}} \right] \left[1 - \frac{\theta + (1-x^\alpha)^\beta}{(1+\theta)(1-x^\alpha)^\beta} e^{-\theta[(1-x^\alpha)^\beta - 1]} \right]^k \quad (38)$$

and

$$f_{n:n}(x) = n \left[\frac{\alpha\beta\theta^2 x^{\alpha-1} e^{-\theta[(1-x^\alpha)^\beta - 1]}}{(1+\theta)(1-x^\alpha)^{2\beta+1}} \right] \left[1 - \frac{\theta + (1-x^\alpha)^\beta}{(1+\theta)(1-x^\alpha)^\beta} e^{-\theta[(1-x^\alpha)^\beta - 1]} \right]^{n-1} \quad (39)$$

5 Maximum Likelihood Estimation of the Unknown Parameters of the OLinKumD

Let X_1, X_2, \dots, X_n be a sample of size 'n' independently and identically distributed random variables from the OLinKumD with unknown parameters, α , β and θ defined previously.

The likelihood function of the OLinKumD using the pdf is given by:

$$L(X | \alpha, \beta, \theta) = \left(\frac{\alpha\beta\theta^2}{(1+\theta)} \right)^n \prod_{i=1}^n \left\{ x_i^{\alpha-1} (1-x_i^\alpha)^{-2\beta-1} e^{\theta[1-(1-x_i^\alpha)^\beta]} \right\} \quad (40)$$

Let the natural logarithm of the likelihood function be, $l = \log L(X | \alpha, \beta, \theta)$, therefore, taking the natural logarithm of the function equation (40) above gives:

$$l = n \log \alpha + n \log \beta + 2n \log \theta - n \log(1+\theta) + (\alpha-1) \sum_{i=1}^n \log x_i - (2\beta+1) \sum_{i=1}^n \log(1-x_i^\alpha) + \theta \sum_{i=1}^n \left[1 - (1-x_i^\alpha)^\beta \right] \quad (41)$$

Differentiating l partially with respect to α , β and θ respectively gives the following results:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log x_i + (2\beta + 1) \sum_{i=1}^n \left\{ \frac{x_i^\alpha \ln x_i}{(1-x_i^\alpha)} \right\} - \theta \sum_{i=1}^n \left\{ (1-x_i^\alpha)^{-\beta-1} x_i^\alpha \ln x_i \right\} \quad (42)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + 2 \sum_{i=1}^n \log(1-x_i^\alpha) + \theta \sum_{i=1}^n \left\{ (1-x_i^\alpha)^{-\beta} \ln(1-x_i^\alpha) \right\} \quad (43)$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + n \log \beta - \sum_{i=1}^n \log \left[\beta + (1-x_i^\theta)^{-\lambda} - 1 \right] \quad (44)$$

Making equation (42), (43) and (44) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters α , β and θ . Bear in mind that it is always hard solving the above equations analytically and hence the “AdequacyModel” package method is used in R software to get the estimates with available datasets.

6 Applications

In this section, an application to a real life data is provided to illustrate the flexibility of the odd Lindley-Kumaraswamy distribution (OLinKumD) introduced in Section 2 in comparison to the Lomax-Kumaraswamy distribution (LomKumD), Transmuted Kumaraswamy distribution (TransKumD), Kumaraswamy-Kumaraswamy distribution (KumKumD and the conventional Kumaraswamy distribution (KumD). The above distributions are selected because of the fact that they are existing extensions of the Kumaraswamy distribution and therefore the comparison will show if the proposed model is an improvement over some existing extensions of the Kumaraswamy distribution or not. The MLEs of the model parameters are determined and some goodness-of-fit statistics for this distribution are compared with other competitive models.

The model selection is carried out based upon the value of the log-likelihood function evaluated at the MLEs (ℓ), Akaike Information Criterion, AIC , Consistent Akaike Information Criterion, $CAIC$, Bayesian Information Criterion, BIC , Hannan Quin Information Criterion, $HQIC$, Anderson-Darling (A^*), Cramèr-Von Mises (W^*) and Kolmogorov-smirnov (K-S) statistics. The details about the statistics A^* , W^* and K-S are discussed in [31]. Meanwhile, the smaller these statistics are, the better the fit of the distribution is. The required computations are carried out using the R package “AdequacyModel” which is freely available from <http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf>.

The real life dataset: This dataset is on shape measurements of 48 rock samples from a petroleum reservoir. This data was extracted from BP research, image analysis by Ronit Katz, u Oxford and has been used for analysis [25]. The summary of the dataset is given in Table 2 as follows:

Table 1. Summary Statistics for the real life dataset

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
48	0.0903	0.1623	0.1988	0.2627	0.2181	0.4641	0.0069	1.1694	1.1099

Table 2. Maximum likelihood parameter estimates for the real life dataset

<i>Distribution</i>	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
OLinKumD	2.332937	3.413559	7.384488	
LomKumD	2.440826	8.407846	7.318735	2.139304
TransKumD	2.073119	9.541583	0.944371	
KumKumD	0.7563927	3.8589464	6.6999813	3.8826105
KumD	1.739625	9.773743		

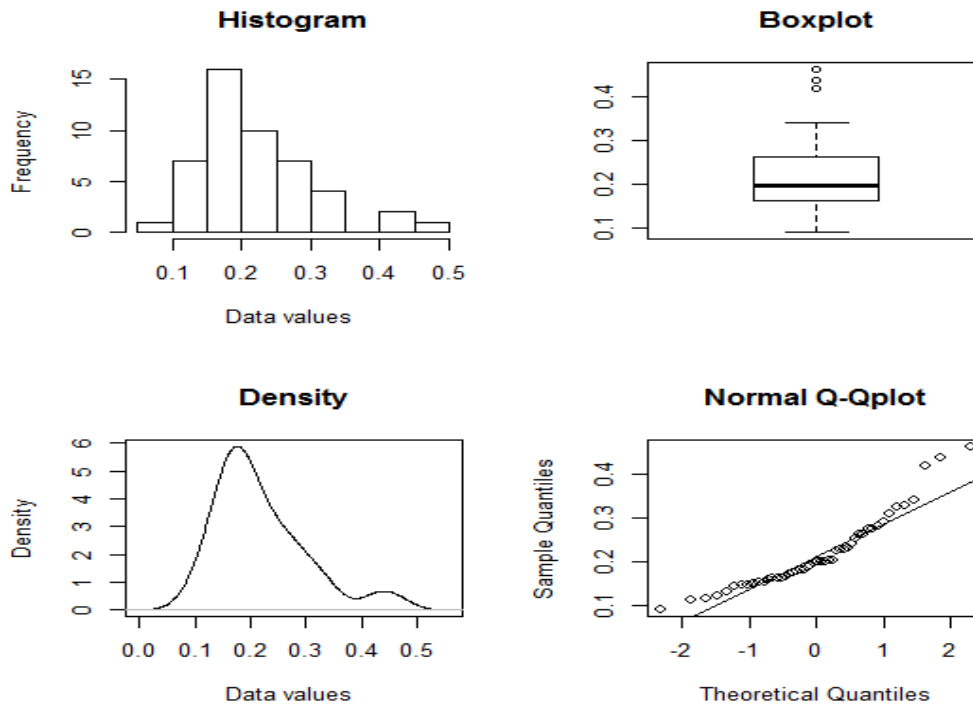


Fig. 5. A graphical summary of the real life dataset

Checking the descriptive statistics in Table 1 and the plots in Fig. 5 above, it is observed that the data takes values between zero and one (1) in accordance with the range for any Kumaraswamy distribution and the dataset is skewed to the right or positively skewed which could be good for models like the proposed OLinKumD.

Table 3. The statistics ℓ , AIC, CAIC, BIC and HQIC for the real life dataset

<i>Distribution</i>	$\hat{\ell}$	<i>AIC</i>	<i>CAIC</i>	<i>BIC</i>	<i>HQIC</i>	<i>Ranks</i>
OLinKumD	-50.72675	-95.4535	-94.90805	-89.8399	-93.33212	2 nd
LomKumD	-51.22048	-94.44096	-93.51073	-86.95616	-91.61244	3 rd
TransKumD	-49.67922	-93.35844	-92.81299	-87.74484	-91.23706	4 th
KumKumD	-55.24402	-102.488	-101.5578	-95.00324	-99.65952	1 st
KumD	-45.08782	-86.17563	-85.90896	-82.43323	-84.76137	5 th

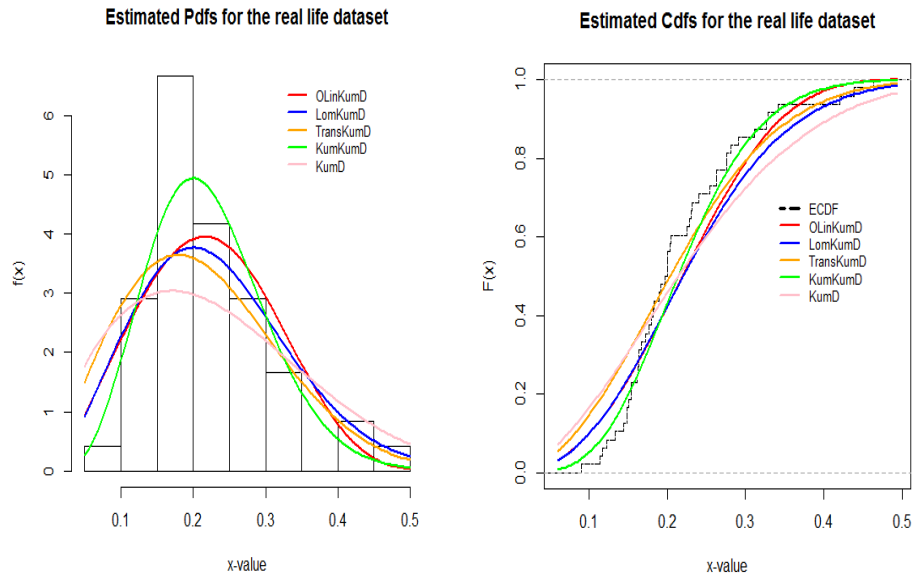


Fig. 6. Histogram and plots of the estimated densities and cdfs of the OLinKumD and other fitted distributions to the real life dataset

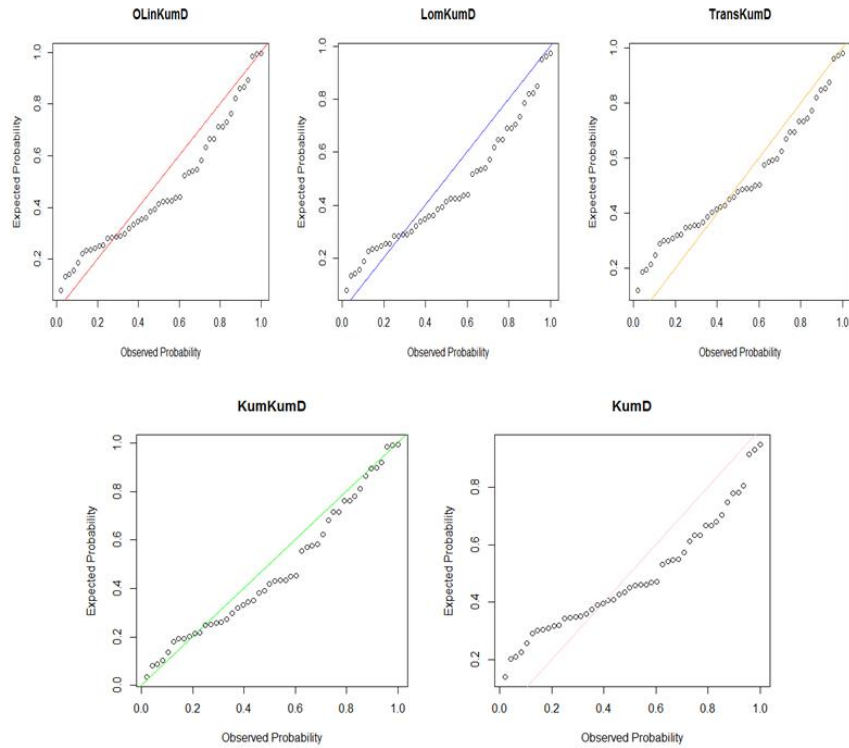


Fig. 7. Probability plots for the fit of the OLinKumD and other competing models based on the real life dataset

Table 4. The A^* , W^* , K-S statistic and P-values for the real life dataset

<i>Distribution</i>	A^*	W^*	<i>K-S</i>	<i>P-Value (K-S)</i>	<i>Ranks</i>
OLinKumD	1.513795	0.2443881	0.164	0.1512	2 nd
LomKumD	0.8902622	0.1471406	0.16464	0.1481	3 rd
TransKumD	0.9787158	0.1606326	0.18409	0.07728	4 th
KumKumD	0.8214046	0.1355988	0.1515	0.2206	1 st
KumD	0.9144384	0.1498356	0.18761	0.06817	5 th

Table 2 lists the values of the MLEs of the model parameters for the datasets, whereas the values of AIC, CAIC, BIC and HQIC are listed in Table 3 and the values of A^* , W^* and K-S for are provided in Table 4.

The plots of the fitted OLinKumD density and cumulative distribution function with those of competing distributions for the dataset are displayed in Fig. 6. The PP-plots of the fitted distributions are also given in Fig. 7 for the real life dataset. From the results in Table 3 and Table 4, it is found that the Kumaraswamy-Kumaraswamy and the odd Lindley-Kumaraswamy distributions are the two best models with better fits to the dataset. The results also show that the two distributions are better compared the three other fitted distributions (transmuted Kumaraswamy, Lomax-Kumaraswamy and the conventional Kumaraswamy distributions). These results are clearly confirmed by the estimated density plots and also the probability plots of the fitted distributions as shown in the figures above.

7 Summary and Conclusion

This article proposed a three-parameter generalization of the Kumaraswamy distribution called “odd Lindley-Kumaraswamy distribution”. The article has studied useful properties of the proposed distribution such as explicit expressions for the moments, generating function, characteristic function, quantile function and related measures, reliability and hazard functions and order statistics. The maximum likelihood method has been used to estimate the parameters of the proposed odd Lindley-Kumaraswamy distribution. A vivid study of the graph of the pdf of the distribution shows that it is flexible and that its shape varies as the values of the parameters are increased or decreased. Also, the plots of the survival and hazard functions of the proposed model indicate that it will be useful for real events where probability of survival decreases with increase in age or time while that of failure increases with time. The proposed distribution and other existing distributions are fitted to a real life dataset to test their flexibility. The results obtained show that the odd Lindley-Kumaraswamy distribution and Kumaraswamy-Kumaraswamy distribution have good fits than the other competing models.

Competing Interests

Authors have declared that no competing interests exist.

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