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Foldness of Positive Implicative Ideals in *BCk*-Algebras

Mahasin A. Ahmed[1*] and Esmat A. Amhed[2]

[1]Department of Mathematics, College of Science, Sudan University of Science and Technology, Khartoum, Republic of Sudan.

[2]IT Department of Electrical Engineering, College of Electrical Engineering, Onaizah Colleges, Kingdom of Saudi Arabia.

Authors' contributions

This work was carried out in collaboration between both authors. Author MAA designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript and managed the analyses of the study. Author EAA managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this article we introduce new notions of (fuzzy) *n*-fold positive implicative ideals, (fuzzy) *n*-fold weak positive implicative ideals, and (fuzzy) n-fold weak implicative (weak) ideals in BCK-algebras and investigate some of their properties.

Keywords: *BCK/BCI algebras; fuzzy BCI - positive implicative ideals of BCI-algebras; fuzzy positive implicative ideal of BCK-algebra; fuzzy point; n-fold positive implicative ideals; n-fold weak positive implicative ideals.*

1 Introduction

The study of BCK/BCI-algebras was initiated by Iséki [1] as generalization of concept of set theoretic difference and propositional calculus, since then a great deal of theorems has been produced on the theory of BCK/BCI-algebras. In(1965) Zadeh [2] was introduced the notion of a fuzzy subset of a set as a method for representing uncertainty .In 1991, Xi [3] defined fuzzy subsets in BCK/BCI-algebras. In 2020 Muhiuddin

**Corresponding author: E-mail: mahasinaliahmed@gmail.com;*

G, Jun YB [4] give further results of neutrosophic subalgebras in BCK/BCI -algebras based on Neutrosophic points.

Huang and Chen [5]. define the notions of *n*-fold implicative ideal and *n*-fold (weak) commutative ideals .Y. B. Jun [6] define an *n*-fold positive implicative, commutative and implicative ideal of BCK-algebra. Muhiuddin G, Kim SJ, Jun YB [7] define Implicative N – ideals of BCK – algebras based on Neutrosophic N – structures.

In the present paper we redefined study the foldness theory of fuzzy positive implicative ideals, positive implicative weak ideals, fuzzy weak positive implicative ideals and weak positive implicative weak ideals in BCK-algebras X . Finally, we construct computer – program for studying foldness theory of positive implicative ideals in BCK-algebra.

2 Preliminaries

2.1 Definition

Iséki K **et al. [1]:** Let X be asset with binary be operation $*$ and a constant 0 .Then $(X;*,0)$ is called a BCI – algebra if it satisfies the following conditions:

For any $x, y, z \in X$

BCI-1. $((x*y)*(x*z))*(z*y)=0$;

BCI-2. $(x*(x*y))*y=0$;

BCI-3. $x*x=0$;

BCI-4. $x*y=0$ and $y*x=0 \Rightarrow x=y$

A BCI-algebras is said to be a BCK-algebra if it satisfies:

BCK-5. $0*x=0$.

A binary relation $\leq$ can be defined by

BCK-6. $x \leq y \Leftrightarrow x*y=0$,

then $(X,\leq)$ is a partially ordered set with least element 0.

The following properties also hold in any BCK-algebra ([8], [9]):

1. $x*0=x$;
2. $x*y=0$ and $y*z=0 \Rightarrow x*z=0$;

3. $x * y = 0 \Rightarrow (x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$;

4. $(x * y) * z = (x * z) * y$;

5. $(x * y) * x = 0$;

6. $x * (x * (x * y)) = x * y$; let $(X, *, 0)$ be a BCK-algebra.

2.2 Definition

(Zadeh [2]). A fuzzy subset of a BCK-algebra X is a function $\mu : X \to [0,1]$.

2.3 Definition

(C. Lele [10]). Let ξ be the family of all fuzzy sets in X. For $x \in X$ and $\lambda \in (0,1], x_\lambda \in \xi$ is a fuzzy point iff

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\tilde{X} = \{x_\lambda : x \in X, \lambda \in (0,1]\}$ the set of all fuzzy points on X and we define a binary operation on $\tilde{X}$ as follows

$$x_\lambda * y_\mu = (x * y)_{\min(\lambda,\mu)}$$

2.4 Remark

(C . Lele [10]), the following conditions hold:

$$\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$$

BCI-1'. $((x_\lambda * y_\mu) * (x_\lambda * z_\alpha)) * (z_\alpha * y_\mu) = 0_{\min(\lambda,\mu,\alpha)}$;

BCI-2'. $(x_\lambda * (x_\lambda * y_\mu)) * y_\mu = 0_{\min(\lambda,\mu)}$;

BCI-3'. $x_\lambda * x_\mu = 0_{\min(\lambda,\mu)}$;

BCK-5'. $0_\mu * x_\lambda = 0_{\min(\lambda,\mu)}$;

2.5 Remark

(C. Lele [10]). The condition BCI-4, is not true in $(\tilde{X}, *)$. So the partial order $\leq$ in $(X, *)$ can not be extended to $(\tilde{X}, *)$.

We can also establish the following conditions $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$:

1'. $x_\lambda * 0_\mu = x_{\min(\lambda,\mu)}$;

2'. $x_\lambda * y_\mu = 0_{\min(\lambda,\mu)}$ and $y_\mu * z_\alpha = 0_{\min(\mu,\alpha)} \Rightarrow x_\lambda * z_\alpha = 0_{\min(\lambda,\alpha)}$;

3'. $x_\lambda * y_\mu = 0_{\min(\lambda,\mu)} \Rightarrow (x_\lambda * z_\alpha) * (y_\mu * z_\alpha) = 0_{\min(\lambda,\mu,\alpha)}$ and $(z_\alpha * y_\mu) * (z_\alpha * x_\lambda) = 0_{\min(\lambda,\mu,\alpha)}$;

4'. $(x_\lambda * y_\mu) * z_\alpha = (x_\lambda * z_\alpha) * y_\mu$;

5'. $(x_\lambda * y_\mu) * x_\lambda = 0_{(\lambda,\mu)}$;

6'. $x_\lambda * (x_\lambda * (x_\lambda * y_\mu)) = x_\lambda * y_\mu$;

We recall that if A is a fuzzy subset of a BCK-algebra X , then we have the following:

$$\tilde{A} = \{x_\lambda \in \tilde{X} : A(x) \geq \lambda \ , \ \lambda \in (0,1]\}. \quad \text{(i)}$$

$$\forall \lambda \in (0,1], \tilde{X}_\lambda = \{x_\lambda : x \in X\}, \text{ and } \tilde{A}_\lambda = \{x_\lambda \in \tilde{X}_\lambda : A(x) \geq \lambda\} \quad \text{(ii)}$$

One can easily check that $(\tilde{X}_\lambda ; * , 0_\lambda)$ is a BCK-algebra.

2.6 Definition

(Isèki [11]). A nonempty subset of BCK-algebra X is called an ideal of X if it satisfies

1. $0 \in I$;
2. $\forall x, y \in X, (x * y \in I \text{ and } y \in I) \Rightarrow x \in I$

2.7 Definition

(Liu called a **and Meng [12]).** A nonempty subset I of BCI-algebra X is BCI- positive implicative ideal if it satisfies:

1. $0 \in I$;
2. $\forall x, y, z \in X, ((x * z) * z) * (y * z) \in I \text{ and } y \in I) \Rightarrow x * z \in I$.

2.8 Definition

(Isèki [11]). A nonempty subset I of BCK-algebra X is said to be a positive implicative ideal if it satisfies

1. $0 \in I$;
2. $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$

2.9 Theorem

(Isèki and Tanaka [1]). Given a non empty subset I of a BCK-algebra X, the following are equivalent :

(a) I is a positive implicative ideal,
(b) I is an ideal and for any x, y in X, $(x*y)*y \in I$ implies $x*y \in I$
(c) I is an ideal, and for any x, y, z in X, $(x*y)*z \in I$ implies $(x*z)*(y*z) \in I$.

2.10 Definition

(Xi Tebu SF et al. **[13]).** A fuzzy subset A of a BCK-algebra X is a fuzzy ideal iff

1. $\forall x \in X \;, A(0) \geq A(x)$;
2. $\forall x, y \in X \;, A(x) \geq \min(A(x * y), A(y))$.

2.11 Definition

(Xi [3]). A fuzzy subset A of a BCK-algebra X is called a fuzzy positive implicative ideal of X if

1. $\forall x \in X \;, A(0) \geq A(x)$;
2. $\forall x, y, z \in X \;, A(x * z) \geq \min(A((x * y) * z), A(y * z))$.

2.12 Definition

(C . Lele, [10]). $\tilde{A}$ is a weak ideal of $\tilde{X}$ if

1. $\forall \nu \in \mathrm{Im}(A)\,;\, 0_\nu \in \tilde{A}$;
2. $\forall x_\lambda, y_\mu \in X$. Such that $x_\lambda * y_\mu \in \tilde{A}$ and $y_\mu \in \tilde{A}$, we have $x_{\min(\lambda,\mu)} \in \tilde{A}$.

2.13 Theorem

(Lele, Wu, Weke, Mamadou and Njock [10]). Suppose that A is a fuzzy subset of a BCK-algebra X, then the following conditions are equivalent:

1. A is a fuzzy ideal ;
2. $\forall x_\lambda, y_\mu \in \tilde{A}$, $(z_\alpha * y_\mu) * x_\lambda = 0_{\min(\lambda,\mu,\alpha)} \Rightarrow z_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$;
3. $\forall t \in (0,1]$, the t-level subset $A^t = \{x \in X : A(x) \geq t\}$ in an ideal when $A^t \neq \phi$;
4. $\tilde{A}$ is a weak ideal.

3 Fuzzy *n*-Fold Positive Implicative Weak Ideals

In the following let $\tilde{X}$ is the set of fuzzy points on BCK-algebra X and $n \in \mathbb{N}$ (where $\mathbb{N}$ the set of all the natural numbers).

And let us denote $(\cdots((x * y) * y) * \cdots) * y$ by $x * y^n$

and $(\cdots((x_{\min(\lambda,\mu)} * 0_\mu) * 0_\mu) * \cdots) * 0_\mu$ by $x_\lambda * y_\mu^n$ (where y and y_μ occurs respectively n times) with $x, y \in X$, $x_\lambda, y_\lambda \in \tilde{X}$.

3.1 Definition

A nonempty subset I of a BCK-algebra X is called an *n*-fold positive implicative ideal of X if it satisfies the following :

1. $0 \in I$;
2. $\forall x, y, z \in X$, .

$((x * y) * z) \in I$ and $y * z \in I) \Rightarrow x * z^n \in I$

3.2 Definition

Let X be a BCK – algebra . A fuzzy subset A of X is said to be a fuzzy n-fold positive implicative ideal of X if it satisfies the following:

1. $\forall x \in X$, $A(0) \geq A(x)$;
2. $\forall x, y, z \in X$,$A(x * z^n) \geq \min(A((x * y) * z)), A(y * z))$.

3.3 Definition

$\tilde{A}$ is a positive implicative weak ideal of $\tilde{X}$ if it satisfies following :

1. $\forall \nu \in \operatorname{Im}(A)$, $0_\nu \in \tilde{A}$;
2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$, if $((x_\lambda * y_\mu) * z_\alpha) \in \tilde{A}$ and $y_\mu * z_\alpha \in \tilde{A}$ we have

$x_{\min(\lambda,\mu)} * z_\alpha \in \tilde{A}$.

3.4 Definition

$\tilde{A}$ is an n- fold a positive implicative weak ideal of $\tilde{X}$ if it satisfies following:

1. $\forall \nu \in \mathrm{Im}(A)$, $0_\nu \in \tilde{A}$;
2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$, if $((x_\lambda * y_\mu) * z_\alpha) \in \tilde{A}$ and $y_\mu * z_\alpha \in \tilde{A}$,then

$x_{\min(\lambda,\mu)} * z^n{}_\alpha \in \tilde{A}$

3.5 Example

Let $X = \{0, a, b, c\}$ be a BCK-algebra with Cayley table as follows:

*	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	c	0

Let A be a fuzzy set in X defined by $A(0) = A(a) = A(b) = 1$ and $A(c) = t$, where $t = [0,1)$. One can easily check that for $n > 2$

$$\tilde{A} = \{0_\lambda : \lambda \in (0,1]\} \cup \{a_\lambda : \lambda \in (0,1]\} \cup \{b_\lambda : \lambda \in (0,1]\} \cup \{c_\lambda : \lambda \in [0,1)\}$$

is an n-fold positive implicative weak ideal.

3.6 Remark

The necessary and sufficient condition for $\tilde{A}$ is to be 1-fold positive implicative weak ideal of a BCK-algebra $\tilde{X}$ is $\tilde{A}$ is a positive implicative weak ideal of $\tilde{X}$.

3.7 Theorem

A fuzzy ideal μ of BCK-algebra X is a fuzzy 1- fold a positive implicative iff

$$\forall x, y, z, \mu(x * z) \geq \mu((x * y) * y) \rightarrow (i)$$

Proof. $(\Rightarrow)$ .Assume that μ a fuzzy 1-fold positive implicative ideal of X and replaced z by y in Definition 3.2 then

$$\mu(x * y) \geq \min(\mu((x * y) * y), \mu(y * y))$$

$$= \min(\mu((x * y) * y), \mu(0))$$

$= \mu((x * y) * y)$, Which proof $(\Rightarrow)$.

For $(\Rightarrow)$ let μ be fuzzy ideal satisfying (i). Since

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$$

And since any fuzzy ideal is order reversing we have

$$\mu((x*y)*z) \leq \mu(((x*z)*z)*(y*z))$$

It follows from Definition 2.8 (2) and (i) that

$$\mu(x*z) \geq \mu((x*z)*z)$$

$$\geq \min\left(\mu(((x*z)*z)(y*z)), \mu(y*z)\right).$$

This completes the proof.

3.8 Proposition

Suppose A is a fuzzy n-fold positive implicative ideal of a BCK-algebra X then

$\forall x_{\lambda}, y_{\mu} \in \tilde{X}$ such that $(x_{\lambda} * y_{\mu}) \in \tilde{A}$, then

$$(x*(x*y)^n)_{\min(\lambda,\mu)} = x_{\min(\lambda,\mu)} * (x*y)^n{}_{\min(\lambda,\mu)} \in \tilde{A}$$

Proof. Let $x_{\lambda}, y_{\mu} \in \tilde{A}$. Since A is a fuzzy n-fold positive implicative ideal, we have

$$A(x*(x*y)^n \geq \min(A((x*y)*(x*y)), A(y*(x*y)))$$

$$= \min(A(0), A(y*(x*y)) = A(y*(x*y)) \geq \min(\lambda,\mu)$$

Therefore $(x*(x*y)^n)_{\min(\lambda,\mu)} = x_{\min(\lambda,\mu)} * (x*y)^n{}_{\min(\lambda,\mu)} \in \tilde{A}$.

3.9 Theorem

The necessary and sufficient condition of a fuzzy subset A of X to be a fuzzy n-fold positive implicative ideal is $\tilde{A}$ is an n-fold positive implicative weak ideal.

Proof. $\Rightarrow$ - Let $\lambda \in \mathrm{Im}(A)$, it is easy to prove that $0_{\lambda} \in \tilde{A}$;

- Let $(x_{\lambda} * y_{\mu}) * z_{\alpha} \in \tilde{A}, and, y_{\mu} * z_{\alpha} \in \tilde{A},$ thus

$A\left((x * y) * z\right) \geq \min(\lambda, \mu, \alpha)$ and $A(y * z) \geq \min(\mu, \alpha)$.

Since A is a fuzzy n-fold positive implicative ideal, we have

$$A\left(x * z^{n}\right) \geq \min(A\left((x * y) * z\right), A(y * z)) \geq \min(\min(\lambda, \mu, \alpha), \min(\mu, \alpha))$$
$$= \min(\lambda, \mu, \alpha).$$

Therefore $\left(x * z^{n}\right)_{\min(\lambda, \mu, \alpha)} = \left(x_{\min(\lambda, \mu)} * z^{n}{}_{\alpha}\right) \in \tilde{A}$.

$\Leftarrow$ - Let $x \in X$, it is easy to prove that $A(0) \geq A(x)$;

- Let $x, y, z \in X$ and let $A((x * y) * z) = \beta$ and $A(y * z) = \alpha$, then

$((x * y) * z)_{\min(\beta, \alpha)} = (x_{\beta} * y_{\alpha}) * z_{\alpha} \in \tilde{A}$ and $y_{\alpha} * z_{\alpha} \in \tilde{A}$.

Since $\tilde{A}$ is n-fold positive implicative weak ideal, we have

$x_{\min(\beta, \alpha)} * z^{n}{}_{\alpha} = (x * z^{n})_{\min(\beta, \alpha)} \in \tilde{A}$.

Thus $A(x * z^{n}) \geq \min(\beta, \alpha) = \min(A((x * y) * z), A(y * z))$ □

3.10 Proposition

An n-fold positive implicative weak ideal is a weak ideal.

Proof. Let $x_{\lambda}, y_{\lambda} \in \tilde{X}$ and $x_{\lambda} * y_{\mu} = (x_{\lambda} * y_{\mu}) * 0_{\mu} \in \tilde{A}, y_{\mu} * 0_{\mu} \in \tilde{A}$

Since $\tilde{A}$ is an n-fold implicative weak ideal, we have

$x_{\min(\lambda, \mu)} = (\cdots((x_{\min(\lambda, \mu)} * 0_{\mu}) * 0_{\mu}) * \cdots) * 0_{\mu} \in \tilde{A}$

3.11 Theorem

Let $\{\tilde{A}_{i \in I}\}$ be a family of *n*-fold positive implicative weak ideals and $\{A_{i \in I}\}$ be a family of fuzzy *n*-fold positive implicative ideals . then (1) $\bigcap_{i \in I} \tilde{A}_{i}$ is an *n*-fold positive implicative weak ideal.

1. $\bigcup_{i\in I}\tilde{A}_i$ is an *n*-fold positive implicative weak ideal.
2. $\bigcap_{i\in I} A_i$ is a fuzzy *n*-fold positive implicative ideal .
3. $\bigcup_{i\in I} A_i$ is a fuzzy *n*-fold positive implicative ideal .

Proof. (1) $\forall\lambda\in \mathrm{Im}\left(\bigcap_{i\in I}\tilde{A}_i\right)$, then $\lambda\in \mathrm{Im}(\tilde{A}_i),\forall i$, so, $0_\lambda\in\tilde{A}_i,\forall i$, i.e. $0_\lambda\in\bigcap_{i\in I}\tilde{A}_i$. For every $x_\mu,y_\lambda,z_\alpha\in\tilde{X}$, if $((x_\lambda * y_\mu)*z_\alpha)\in\bigcap_{i\in I}\tilde{A}_i$ *and* $(y_\mu * z_\alpha)\in\bigcap_{i\in I}\tilde{A}_i$,then

$((x_\lambda * y_\mu)*z_\alpha)\in\tilde{A}_i$ *and* $y_\mu * z_\alpha\in\tilde{A}_i\ \forall i$, thus

$x_{\min(\lambda,\mu)} * z^n{}_\alpha\in\tilde{A}_i\ \forall i$

So $x_{\min(\lambda,\mu)} * z_\alpha{}^n\in\bigcap_{i\in I}\tilde{A}_i$. Thus $\bigcap_{i\in I}\tilde{A}_i$ is an *n*-fold implicative weak ideals .

(2) . (1) $\forall\lambda\in \mathrm{Im}\left(\bigcup_{i\in I}\tilde{A}_i\right)$, then $\exists i_0\in I, such, that\ \lambda\in\tilde{A}_{i_0}$, so, $0_\lambda\in\tilde{A}_{i_0}$, i.e. $0_\lambda\in\bigcup_{i\in I}\tilde{A}_i$. For every $x_\mu,y_\lambda,z_\alpha\in\tilde{X}$, if $((x_\lambda * y_\mu)*z_\alpha)\in\bigcup_{i\in I}\tilde{A}_i$ *and* $(y_\mu * z_\alpha)\in\bigcup_{i\in I}\tilde{A}_i$,then $\exists i_0\in I$ such that

$((x_\lambda * y_\mu)*z_\alpha)\in\tilde{A}_{i_0}$ *and* $y_\mu * z_\alpha\in\tilde{A}_{i_0}\ \forall i$, thus $x_{\min(\lambda,\mu)} * z^n{}_\alpha\in\tilde{A}_{i_0}$

So $x_{\min(\lambda,\mu)} * z^n{}_\alpha\in\bigcup_{i\in I}\tilde{A}_i$. Thus $\bigcup_{i\in I}\tilde{A}_i$ is an *n*-fold implicative weak ideals.

(3) Follows from (1) and Theorem 3.8.
(4) Follows from (2) and Theorem 3.8.

4 Fuzzy n-Fold Weak Positive Implicative Ideals

In this section, we define and give some characterizations of (fuzzy) n-fold weak implicative(weak) ideals in BCK-algebras.

4.1 Definition

A nonempty subset I of X is called an n-fold weak positive implicative ideal of X if it satisfies

1. $0\in I$;
2. $\forall x,y,z\in X\ ,\left(\left(x*z\right)*z^n\right)*\left(y*z\right)\in I, and, y\in I\ \Rightarrow x*z\in I$

4.2 Definition

Let X be a BCK – algebra . A fuzzy subset A of X is said to a fuzzy n-fold weak positive implicative ideal of X if it satisfies the following:

1. $\forall x \in X \ , A(0) \geq A(x)$;
2. $\forall x, y \in X \ , A(x*z) \geq \min\left(A\left(\left(\left(x*z\right)*z^{n}\right)*\left(y*z\right)\right), A\left(y\right)\right)$

4.3 Definition

$\tilde{A}$ is a weak positive implicative weak ideal of $\tilde{X}$ if it satisfies following : 1. $\forall \nu \in \operatorname{Im}(A)\ , 0_{\nu} \in \tilde{A}$;

2. $\forall x_{\lambda}, y_{\mu}, z_{\alpha} \in \tilde{X}$

$(((x_{\lambda}*z_{\alpha})*z_{\alpha})*(y_{\mu}*z_{\alpha})) \in \tilde{A}$ and $y_{\mu} \in \tilde{A}) \Rightarrow (x_{\min(\lambda,\mu)}*z_{\alpha}) \in \tilde{A}$.

4.4 Definition

$\tilde{A}$ is an n-fold an weak positive implicative weak ideal of $\tilde{X}$ if it satisfies following : 1. $\forall \nu \in \operatorname{Im}(A)\ , 0_{\nu} \in \tilde{A}$;

2. $\forall x_{\lambda}, y_{\mu}, z_{\alpha} \in \tilde{X}$,

$(((x_{\lambda}*z_{\alpha})*z^{n}_{\alpha})*(y_{\mu}*z_{\alpha})) \in \tilde{A}$ and $y_{\mu} \in \tilde{A}) \Rightarrow (x_{\min(\lambda,\mu)}*z_{\alpha}) \in \tilde{A}$.

4.5 Example

Let $X = \{0,1\}$ in which $*$ is given by

$1*0=1$ and $0*0=0*1=1*1=0$

Then $(X\ ;*, 0)$ is a BCK-algebra. Let $t_1, t_2 \in (0,1]$ and let us define a fuzzy subset $A : X \to [0,1]$ by

$t_1 = A(0) > A(1) = t_2$

It is easy to check that for any $n > 2$

$\tilde{A} = \{0_{\lambda} : \lambda \in (0, t_1]\} \cup \{I_{\lambda} : \lambda \in (0, t_2]\}$

Is an n-fold weak positive implicative weak ideal.

4.6 Remark

The necessary and sufficient condition for $\tilde{A}$ to be a 1-fold weak implicative positive weak ideal of a BCK-algebra X is $\tilde{A}$ is a weak positive implicative weak ideal.

4.7 Theorem

An n-fold weak positive implicative weak ideal is a weak ideal.

Proof. By setting $z_\alpha = 0$ in Definition 4.4. one obtain that $\forall x_\lambda, y_\mu \in \tilde{X}$

$((x_\lambda * y_\mu) \in \tilde{A}$ and $y_\mu \in \tilde{A}) \Rightarrow x_{\min(\lambda,\mu)} \in \tilde{A}$.

This shows that is a weak ideal ,proving the Theorem.

4.8 Theorem

a fuzzy n-fold weak positive implicative ideal is a fuzzy ideal.

Proof. By setting $z = 0$ in Definition 4.4. one obtain that

$$\forall x, y, z \in X \ , A(x) \geq \min(A(x * y), A(y))$$

This shows that is a fuzzy ideal ,proving the Theorem.

4.9 Theorem

The necessary and sufficient condition for a fuzzy subset A of X to be a fuzzy n-fold weak positive implicative ideal is $\tilde{A}$ is an n-fold weak positive implicative weak ideal.

Proof. $\Rightarrow$ - Let $\lambda \in \operatorname{Im}(A)$ obviously $0_\lambda \in \tilde{A}$;

- Let $(((x_\lambda * z_\alpha) * z^n{}_\alpha) * (y_\mu * z_\alpha)) \in \tilde{A}$ and $y_\mu \in \tilde{A}$, then

$$A(((x_\lambda * z_\alpha) * z^n{}_\alpha) * (y_\mu * z_\alpha)) \geq \min(\lambda, \mu, \alpha) \text{ and } A(y) \geq \mu .$$

Since A is a fuzzy n-fold weak implicative ideal, we have

$$A(x * z) \geq \min(A(((x * z) * z^n) * (y * z)), A(y))$$

$$\geq \min(\min(\lambda, \mu, \alpha), \mu) = \min(\lambda, \mu, \alpha) .$$

Therefore $(x*z)_{\min(\lambda,\mu,\alpha)} = x_{\min(\lambda,\mu)} * z_{\alpha} \in \tilde{A}$..

$\Leftarrow$ - Let $x \in X$, it is easy to prove that $A(0) \geq A(x)$;

- Let $x, y, z \in X$,$A(((x*y)*z^{n}))*(y*z)) = \beta$ and $A(y) = \alpha$.

Then $((x*(x*y^{n}))*z)_{\min(\beta,\alpha)} = (x_{\beta}*(x_{\beta}*y^{n}_{\beta}))*z_{\alpha} \in \tilde{A}$ and $z_{\alpha} \in \tilde{A}$.

Since $\tilde{A}$ is n-fold weak implicative weak ideal, we have

$x_{\min(\beta,\alpha)} * z_{\beta}) = (x*z)_{\min(\beta,\alpha)} \in \tilde{A}$

Hence $A(x*z) \geq \min(\beta,\alpha) = \min\left(\left(\left((x*z)*z^{n}\right)*(y*z)\right), A(y)\right)$□

4.10 Theorem

If A is a fuzzy n-fold weak positive implicative ideal; then

$\forall x_{\lambda}, z_{\alpha} \in \tilde{X}$ such that $\left(\left(x_{\min(\mu,\lambda)} * z_{\alpha}\right) * z_{\alpha}\right) \in \tilde{A}$, we have

$x_{\min(\lambda,\mu)} * z_{\alpha} \in \tilde{A}$;

Proof. Let $\left(\left(x_{\min(\mu,\lambda)} * z_{\alpha}\right) * z_{\alpha}\right) \in \tilde{A}$,. Since A is a fuzzy n-fold weak positive implicative ideal, we have

$A(x*z) \geq \min(A((x*z)*z^{n}))) , A(0))$

$= A((x*z)*z^{n}) \geq \min(\lambda,\alpha)$.

Therefore $(x*z)_{\min(\lambda,\alpha)} = x_{\min(\lambda,\mu)} * z_{\alpha} \in \tilde{A}$.

4.11 Theorem

Let $\{\tilde{A}_{i\in I}\}$ be a family of *n*-fold weak positive implicative weak ideals and $\{A_{i\in I}\}$ be a family of fuzzy *n*-fold weak positive implicative ideals . then (1) $\bigcap_{i\in I} \tilde{A}_{i}$ is an *n*-fold weak positive implicative weak ideal.

(2)) $\bigcup_{i\in I}\tilde{A}_i$ is an *n*-fold weak positive implicative weak ideal.
(3)) $\bigcap_{i\in I} A_i$ is a fuzzy *n*-fold weak positive implicative ideal.
(4)) $\bigcup_{i\in I} A_i$ is a fuzzy *n*-fold weak positive implicative ideal.

Proof. (1) $\forall\lambda\in \mathrm{Im}\left(\bigcap_{i\in I}\tilde{A}_i\right)$, then $\lambda\in\mathrm{Im}(\tilde{A}_i),\forall i$, so, $0_\lambda\in\tilde{A}_i,\forall i$, i.e. $0_\lambda\in\bigcap_{i\in I}\tilde{A}_i$. For every $x_\mu,y_\lambda,z_\alpha\in\tilde{X}$, if $(((x_\lambda * z_\alpha)*z_\alpha{}^n)*(y_\mu * z_\alpha))\in\bigcap_{i\in I}\tilde{A}_i\ and, y_\mu\in\bigcap_{i\in I}\tilde{A}_i$,then

$(((x_\lambda * z_\alpha)*z_\alpha{}^n)*(y_\mu * z_\alpha))\in\tilde{A}_i\ and, y_\mu\in\tilde{A}_i\ \forall i$, thus

$x_{\min(\lambda,\mu)} * z_\alpha\in\tilde{A}_i\ \forall i$

So $x_{\min(\lambda,\mu)} * z_\alpha\in\bigcap_{i\in I}\tilde{A}_i$. Thus $\bigcap_{i\in I}\tilde{A}_i$ is an *n*-fold weak implicative weak ideals.

(2). (1) $\forall\lambda\in\mathrm{Im}\left(\bigcup_{i\in I}\tilde{A}_i\right)$, then $\exists i_0\in I, such, that\ \lambda\in\tilde{A}_{i_0}$, so, $0_\lambda\in\tilde{A}_{i_0}$, i.e. $0_\lambda\in\bigcup_{i\in I}\tilde{A}_i$. For every $x_\mu,y_\lambda,z_\alpha\in\tilde{X}$, if $(((x_\lambda * z_\alpha)*z_\alpha{}^n)*(y_\mu * z_\alpha))\in\bigcup_{i\in I}\tilde{A}_i\ and, y_\mu\in\bigcup_{i\in I}\tilde{A}_i$,then $\exists i_0\in I$ such that

$(((x_\lambda * z_\alpha)*z_\alpha{}^n)*(y_\mu * z_\alpha))\in\tilde{A}_{i_0}\ and, y_\mu\in\tilde{A}_{i_0}\ \forall i$, thus $x_{\min(\lambda,\mu)} * z_\alpha\in\tilde{A}_{i_0}$

So $x_{\min(\lambda,\mu)} * z_\alpha\in\bigcup_{i\in I}\tilde{A}_i$. Thus $\bigcup_{i\in I}\tilde{A}_i$ is an *n*-fold weak implicative weak ideals.

(3) Follows from (1) and Theorem 3.8.
(4) Follows from (2) and Theorem 3.8.

5 Algorithms

Here We Give Some Algorithms For Studding The Structure Of The Foldness Of Fuzzy positive Implicative Ideals In BCK-Algebras

ALGORITHM FOR POSITIVE IMPLICATIVE IDEALS OF BCI-ALGEBRA

Input (X :*BCK-algebra*, $*$: binary operation, I : subset of X) ;
Output(" I is a BCI - positive implicative ideal of X or not");
Begin
If $I=\phi$ then

```
   go to (1.);
 EndIf
 If 0 ∉ I then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not (Stop) do
   j := 1;
  While j ≤ |X| and not (Stop) do
    k := 1;
    While k ≤ |X| and not (Stop) do
     If ((x_i * z_k) * z_k * (y_i * z_k)) ∈ I and y_j ∈ I then
      If x_i * z_k ∉ I
       Stop:=true;
      EndIf
     EndIf
    Endwhile
   Endwhile
  Endwhile
  If Stop then
   Output ("I is a positive implicative ideal of X")
  Else
   (1.) Output ("I is not a positive implicative ideal of X")
 EndIf
End
```

ALGORITHM FOR POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRA

```
Input (X :BCK-algebra, * : binary operation, I : subset of X);
Output("I is a positive implicative ideal of X or not");
Begin
 If I = φ then
   go to (1.);
 EndIf
 If 0 ∉ I then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not (Stop) do
   j := 1;
  While j ≤ |X| and not (Stop) do
    k := 1;
```

```
      While $k \le |X|$ and not (Stop) do
       If $(x_i * y_j) * z_k \in I$ and $y_j * z_k \in I$ then
        If $x_i * z_k \notin I$
         Stop:=true;
        EndIf
       EndIf
      Endwhile
     Endwhile
    Endwhile
    If Stop then
     Output ("$I$ is a positive implicative ideal of $X$")
    Else
     (1.) Output ("$I$ is not a positive implicative ideal of $X$")
    EndIf
  End
```

ALGORITHM FOR FUZZY POSITIVE IMPLICATIVE IDEALS OF BCI-ALGEBRA

```
Input ($X$ :BCK-algebra, $*$ : binary operation, $A$ : fuzzy subset of $X$) ;
Output("$A$ is a fuzzy BCI - positive implicative ideal of $X$ or not");
Begin
  Stop:=false;
  $i := 1$;
  While $i \le |X|$ and not (Stop) do
    If $A(0) < A(x_i)$ then
      Stop:=true;
    EndIf
    $j := 1$;
    While $j \le |X|$ and not (Stop) do
      $k := 1$;
      While $k \le |X|$ and not (Stop) do
       If $A(x * z) < Min(A((x * z) * z) * (y * z)), A(y))$ then
        Stop:=true;
       EndIf
      Endwhile
     Endwhile
    Endwhile
   If Stop then
    Output ("$A$ is not a fuzzy positive implicative ideal of $X$")
   Else
    Output ("$A$ is a fuzzy positive implicative ideal of $X$")
   EndIf
End
```

ALGORITHM FOR FUZZY POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRA

```
Input ( X :BCK-algebra, * : binary operation, A : fuzzy subset of X ) ;
Output(" A is a fuzzy positive implicative ideal of X or not");
Begin
 Stop:=false;
 i := 1;
 While i ≤ |X| and not (Stop) do
   If A(0) < A(x_i) then
     Stop:=true;
   EndIf
    j := 1;
   While j ≤ |X| and not (Stop) do
     k := 1;
     While k ≤ |X| and not (Stop) do
A(x * z) < Min(A((x * y) * z), A(y * z))
        Stop:=true;
       EndIf
      Endwhile
     Endwhile
   Endwhile
  If Stop then
   Output (" A is not a fuzzy positive implicative ideal of X ")
  Else
   Output (" A is a fuzzy positive implicative ideal of X ")
  EndIf
End
```

ALGORITHM FOR N-FOLD POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRA

```
Input( X :BCK-algebra, I : subset of X, n ∈ ℕ );
Output(" I is an n-fold positive implicative ideal of X or not");
Begin
 If I = ϕ then
   go to (1.);
 EndIf
 If 0 ∉ I then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not (Stop) do
   j := 1;
  While j ≤ |X| and not (Stop) do
    k := 1;
```

While $k \le |X|$ and not (*Stop*) do
If $(x_i * y_j) * z_k \in I$ and $y_j * z_k \in I$ then
If $x_i * z^n{}_k \notin I$
Stop:=true;
EndIf
EndIf
Endwhile
Endwhile
Endwhile
If *Stop* then
Output ("I is an *n*-fold positive implicative ideal of X")
Else
(1.) Output ("I is not an *n*-fold i positive implicative ideal of X")
EndIf
End

ALGORITHM FOR FUZZY *N* -FOLD POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRA

Input(X :*BCK-algebra*, $*$: binary operation, A fuzzy subset of X);
Output("A is a fuzzy *n*-fold positive implicative ideal of X or not");
Begin
Stop:=false;
$i := 1$;
While $i \le |X|$ and not (*Stop*) do
If $A(0) < A(x_i)$ then
Stop:=true;
EndIf
$j := 1$;
While $j \le |X|$ and not (*Stop*) do
$k := 1$;
While $k \le |X|$ and not (*Stop*) do
If $A(x * z^n) < Min(A((x * y) * z), A(y * z))$ then
Stop:=true;
EndIf
Endwhile
Endwhile
Endwhile
If *Stop* then
Output ("A is not a fuzzy *n*-fold positive implicative ideal of X")
Else
Output ("A is a fuzzy *n*-fold positive implicative ideal of X")
EndIf
End

ALGORITHM FOR *N*-FOLD WEAK POSITIVE IMPLICATIVE IDEALS

```
Input( X :BCK-algebra, I : subset of X, n ∈ ℕ );
Output(" I is an n-fold weak positive implicative e ideal of X or not");
Begin
 If I = ϕ then
   go to (1.);
 EndIf
 If 0 ∉ I then
   go to (1.);
 EndIf
 Stop:=false;
 i := 1;
 While i ≤ |X| and not (Stop) do
   j := 1;
   While j ≤ |X| and not (Stop) do
     k := 1;
     While k ≤ |X| and not (Stop) do
      If (((x_i * z_k) * z_k^n) * (y_j * z_k)) ∈ I and y_j ∈ I then
       If x_i * z_k ∉ I then
        Stop:=true;
       EndIf
      EndIf
     Endwhile
    Endwhile
  Endwhile
  If Stop then
   Output (" I is an n-fold weak positive implicative ideal of X ")
  Else
  (1.) Output (" I is not an n-fold weak positive implicative ideal of X ")
  EndIf
End
```

ALGORITHM FOR FUZZY *N*-FOLD WEAK POSITIVE IMPLICATIVE IDEALS

```
Input( X :BCK-algebra, * : binary operation, A : fuzzy subset of X );
Output(" A is a fuzzy n-fold weak positive implicative ideal of X or not");
Begin
 Stop:=false;
 i := 1;
  While i ≤ |X| and not (Stop) do
   If A(0) < A(x_i) then
    Stop:=true;
   EndIf
    j := 1;
```

While $j \leq |X|$ and not (*Stop*) do
$k := 1$;
While $k \leq |X|$ and not (*Stop*) do
If $A(x_i * z_k) < Min(A(((x_i * z_k) * z^n_k) * (y_j * z_k)), A(y_j))$ then
Stop:=true;
EndIf
Endwhile
Endwhile
Endwhile
If *Stop* then
Output (" A is not a fuzzy *n*-fold weak positive implicative ideal of X ")
Else
Output (" A is a fuzzy *n*-fold weak positive implicative ideal of X ")
EndIf
End

6 Conclusion and Future Research

In this paper we introduce new notions of (fuzzy) *n*-fold positive implicative ideals, and (fuzzy) *n*-fold weak positive implicative ideals in BCK-algebras,Then we studied relationships between different type of n- fold positive implicative ideals and investigate several properties of foldness theory of positive implicative ideals in BCK-algebras. Finally, we construct some algorithms for studying foldness theory of positive implicative ideals in BCK-algebras.

In our future study of foldness ideals in BCK/BCI algebras ,may be the following topics should be considered:

(1) developing the properties of foldness of positive implicative ideals of BCK/BCI algebras.
(2) finding useful results on other structures of foldness theory of ideals of BCK/BCI algebras.
(3) constructing the related logical properties of such structures.
(4) one may also apply this concept to study some applications in many fields like decision making ,knowledge base systems ,medical diagnosis ,data analysis and graph theory.

Competing Interests

Authors have declared that no competing interests exist.

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