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Rigidity and Shrinkability of Diagonalizable Matrices

Dimitris Karayannakis[1*] and Maria-Evgenia Xezonaki[2]

[1]Department of Informatics Engineering, TEI of Crete, Heraklion 71004, Greece.
[2]Department of Informatics and Telecommunications, University of Athens, Athens 157 84, Greece.

Authors' contributions

This work was carried out in collaboration between the two authors. Author DK designed the study, wrote the theoretical part, wrote the initial draft of the manuscript and managed literature searches. Author MEX managed the matlab calculation whenever necessary for some examples, double- checked the validity of the theoretical predictions and added some new examples of her own. Both authors read and approved the final draft of the manuscript.

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Abstract

We introduce the seemingly new concept of a rigid matrix based on the comparison of its sparsity to the sparsity of the natural powers of the matrix. Our results could be useful as a usage guide in the scheduling of various iterative algorithms that appear in numerical linear algebra. Especially in Sparse matrix-vector multiplication and they could also be used in matrix norm error analysis.

Keywords: Sparse matrix; diagonalizable matrix; matricial limits; matrix norms.

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Introduction

Any algorithm intended to be implemented on a computer has a computational cost that depends upon the amount of resources that will be used. For simplicity here the terms cost and complexity load, refer only

**Corresponding author: E-mail: dkar@ie.teicrete.gr, dkar@staff.teicrete.gr;*

to the number of floating point operations (additions, subtractions, multiplication and divisions) performed during one run of the algorithm, together with a count of the number of square roots; note that we do not taking into account memory usage. Then we could predict cost and load before a chain of algorithmic computations have taken place, but only ideally since, in reality, we obtain results that are not accurately achievable. In this paper we do not attempt to calculate a cost and all samples data is given in the form of square matrices preferably diagonalizable. By asymptotic computation cost of an algorithm, we refer to the behavior of the cost C(n) for $n \rightarrow +\infty$. Note also that the case of the "big O" description is closely related except that in our cases we consider the order of magnitude of small quantities while our concern with cost leads to consideration of the order of magnitude of large quantities, Finally we would like to emphasize that

(i) Whenever the spectrum of a matrix is to be calculated using a PC program one has to take into account the sensitivity of eigenvalues to tiny perturbations.
(ii) There exist many situations in scientific and engineering computations that cannot be comprehended and/or explained by using a single number though, when a decision is needed, it often amounts to distilling one number out of many (see e.g. [1]).
(iii) When we attempt iterative refinements the effort could be in vain since we may encounter too ill-conditioned matrices within too few rounding errors or plainly inaccurate from the start.
(iv) In all cases, trial and error is the best guidance for the "ideal time to terminate" an approximating sequence of matrices ,independently if the calculations are performed by hand or using a scientific PC programs (like in this paper that we use the matlab).

Chapter 1. Shrincable Matrices

We will focus exclusively upon diagonalizable (mainly but not necessarily real), matrices the set of which for nxn matrices we will denote by M_n and in order to avoid trivialities we will exclude the zero matrix. It is well known that the concept of sparsity of a matrix refers to the presence of zero entries ; in order to characterize as sparse a matrix we usually demand much more than half of its elements to be zero. Sparse matrices play a key role in numerical linear algebra (see e.g. [2,3] and [4]) and in recent years have many pleasant and sometimes even unexpected applications (see e.g [5], or [6]).

We engineer the following definition which is purely technical in order to facilitate the introduction of the new concepts to be presented later on. Note that Def, 1.1 could had been applied to any square matrix and that Def. 1.2 can be extended even to non diagonalizable square matrices under weaker conditions but we will not pursue these issues.

Definition 1.1: A square matrix with n^2 elements will be called **m/ n^2- sparse**,$1 \leq m \leq n^2-1$, whenever m of its elements are zero.

Definition 1.2: For $A \in M_n$, A will be called **m/ n^2 shrinkable** if the matrix $\lim_{\kappa\to\infty} A^{\kappa}$ exists (pointwisely) and is m/ n^2 sparse. In particular, if $\lim_{\kappa\to\infty} A^{\kappa}$ is the zero matrix we call A **totally shrinkable**.

Definition 1.3: For $A \in M_n$, A will be called **rigid** if the matrix $\lim_{\kappa\to\infty} A^{\kappa}$ exists and has no zero elements at all;in particular, if A is such that A^{κ} has no zero elements **for all** natural numbers κ we call A **totally rigid**.

Note that im Chapter 3 we examine the rigidness of the DFT matrix.

Remarks 1.1:

(i) It is rather straightforward to conclude that if the absolute values of the n eigenvalues of a nxn matrix are in [0,1] and the eigenvalues are all distinct (which is a sufficient but not necessary condition so that A∈ M_n) , then A is totally shrinkable.

(ii) On the other hand it is obvious that for the nxn matrix A=(a_{ij}) with a_{ij} =1, A^{κ} =(b_{ij}) where b_{ij} $=n^{\kappa-1}$, has no zero entries. We can also easily check that spA consists of the eigen value 0 with algebraic and geometric multiplicity n-1 and also of the eigenvalue n and so it is diagonalizable .Thus A is a totally rigid matrix.

(iii) Examples like the above may have an independent interest but they do not offer any promising use of the concepts of a rigid or a shrinkable. matrix. The utility of these concepts will be discussed in Chapter 2 (where a new concept , that of the **index of rigidity** will be introduced) and mainly in Chapter 4 when we will attempt to connect the estimation of the induced Euclidean norm of powers of A∈ M_n with a possible shrinkable matrix structure.

We will presemt below six examples: the first is a general one classifying the matrices in M_2 according to shrincability and/or rigidness. The other two being numerical examples that we wanted to carry out all necessary calculations without using any PC program, will be restricted within the low size case of n=3. The fourth is a nxn example, for any n≥2 and it is the motivation to introduce the concept of the index of rigidness presented in Chapter 2:

Example 1.1: Let A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real) matrix. Set Δ=(a-b)²+4cd.

(i) If Δ>0 then evidently A∈ M_2 . If in addition $\left|trA \pm \sqrt{\Delta}\right|$ <1 then A is totally shrinkable.

(ii) If Δ=0 and in addition either b and c≠0 A is not diagonalizable and we will not pursue any classification within our approach. If on the other hand a=d and either b or c=0 once again A is not in M_2 .Finally if b=c=0 and $|a|$ <1 then A is totally shrinkable, while for a≤-1 or a>1 $\lim_{\kappa\to\infty} A^{\kappa}$ does not exist and of course for a=1 A=I which is trivially ½ shrinkable.

(iii) If Δ<0 then once more A∈ M_2 . If in addition a²+d²+2bc<1 then A is totally shrinkable and in case that a²+d²+2bc=1 then $\lim_{\kappa\to\infty} A^{\kappa}$ does not exist.

Example 1.2: The matrix A= $\begin{pmatrix} 1 & -3/2 & 3/2 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$ is evidently 4/9 sparse and spA ={-1/2. -1/2, 1}.

It is easy to check that the multiple eigenvalue -1/2 has geometric multiplicity 2 and that a pair of its linearly independent eigenvectors cossists of $(1,0,-1)^t$ and $(0,1,1)^t$ .It is also easy to check that $(1,0,0)^t$ is an eigenvector for 1.

Then via the diagonalization A=EDE^{-1}, where

E= $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$ D= $\begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and E^{-1}= $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$, we obtain that B= $\lim_{\kappa\to\infty} A^{\kappa}$ exists and

B= $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which is 2/3 sparse. Thus A is 2/3 shrincable.

Example 1.3: The matrix A= $\begin{pmatrix} \lambda & 0 & 1-\lambda \\ 0 & \lambda & 1-\lambda \\ 0 & 0 & 1 \end{pmatrix}$, where $|\lambda|<1$ is 4/9 sparse for $\lambda\neq0$.and 7/9sparse for $\lambda=0$, so let us focus ovly on the family A(λ) when the parameter λ (real or not) is mot zero.

It is easy to check that the geometric multiplicity of λ also 2 and then we have the diagonalization A=EDE^{-1},

where E= $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ D= $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and E^{-1}= $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

We thus obtain

$\lim_{\kappa\to\infty} A^{\kappa} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ which is a 2/3 sparse matrix. Note that the result is independe3nt of the parameter λ, something that we will be useful for us in the context of Chapter 1.

Example 1.4: The matrix A= $\begin{pmatrix} \lambda & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $|\lambda|<1$ (λ not necessarily real), is 4/9 sparse for $\lambda\neq0$. Evidently we have spA={λ,. 1, 1}. It is again easy to check that the multiple eigenvalue 1 has geometric multiplicity 2 and that a pair of its linearly independent eigenvectors cosista of $(1,0,1-\lambda)^t$ and $(0,1,1)^t$ .It is also easy to check that $(1,0,0)^t$ .is an eigenvector for λ.

Then via the diagonalization A=EDE^{-1}, where E= $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1-\lambda & -1 & 0 \end{pmatrix}$, D= $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

E^{-1}= $\frac{1}{\lambda-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1-\lambda & -1 & 0 \end{pmatrix}$, we obtain that B= $\lim_{\kappa\to\infty} A^{\kappa}$ exists and B= $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Thus A is 7/9 shrincable

Example 1.5: Let z_m , $1\le m\le n$ be the n roots of unity, as they were introduced by Euler i.e. $z_m = e^{2\pi mi/n}$.

We set A the following nxn upper triangular matrix for $n\ge 2$:

A= $\begin{pmatrix} z_1 & 1 & 0 & ... & 0 \\ 0 & z_2 & 1 & 0 & 0 \\ ... & ... & z_m & ... & 0 \\ 0 & 0 & ... & z_{n-1} & 1 \\ 0 & 0 & 0 & ... & z_n \end{pmatrix}$ which is diagonalizable since its spectrum $\{z_m ,1\le m\le n\}$ consists of distinct numbers (note that $z_n =1$). The diagomalization $A=EDE^{-1}$, where E is the eigenvectors matrix and the diagonal elements of D are the n roots of unity lead to $A^n =E E^{-1} =I$ Thus for $\kappa=\rho n+\upsilon$, $0\le \upsilon<n$,$A^\kappa = A^\upsilon$ and $\lim_{\kappa\to\infty} A^\kappa$ does not exist. and the maximum index of sparsity of A^κ is (n-1)/n Nevertheless an interesting feature of A which we will examone among others in Chapter 2 is estimating t the maximum sparsity of A^υ.

Example 1.6: Let A= $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \zeta \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$ with ζ any parameter. Since spA={1/2.1/2,1,1} and the geometric multiplicity of the eigenvalues 1 and ½ is 2 we know that A is diagonalizable and $A=EDE^{-1}$. where D=diag(1,1,1/2,1/2) and the corresponding eigenvectors matrix is

E= $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2\zeta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ .We find also that $E^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and finally we obtain that

$\lim_{\kappa\to\infty} A^\kappa = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is a 7/8-sparse matrix free of the parameter something worth revisiting when we will examine norm approximations in Chapter 4.

Remark 1.2:

Foe $A\in M_n$ the condition $|spA|<1$ produces a rather narrow family of shrinkable matrices. On the other hand if $A\in M_n$ and $|spA| \cap (1, +\infty) \neq \varnothing$ then evidently $\lim_{\kappa\to\infty} A^\kappa$ will not exist. So it is natural to slightly relax the definition by allowing $1\in spA$ possibly with an algebraic multiplicity equal to its geometric multiplicity (though for very small n 's it is a rather descent exercise to see that we do not have the case described in Def.1.4).The definition that fpllows substantially useful as we will see later on:

Definition 1.4: Foe $A \in M_n$ we call A **almost shrinkable** if and ony if $\lim_{\kappa\to\infty} A^\kappa$ is (n^2 -1)/ n^2 shrinkable.

Example 1.7: Let A= $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & \zeta \\ 0 & 0 & 1/3 \end{pmatrix}$ with ζ any parameter $\neq 0$. Since spA={1/3.1/2,1}.A is diagonalizable. Then A= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1/6\zeta \end{pmatrix}$ diag(1,1/2.1/3) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1/6\zeta \end{pmatrix}$, and we obtain $\lim_{\kappa\to\infty} A^\kappa = \frac{1}{\zeta}$ G, where G=(g_{ij}) is the 3x3 matrix with g_{11} =1and g_{ij} =0 for (i,j)$\neq$(1,1). Thus A is almost shrinkable.

Chapter 2. Index of Rigidity

We introduce now the following additional concepts already announced in Chapter 1;

Definition 2.1: Let A be a non zero (not necessarily diagonalizable) nxn matrix that is not totally rigid. We call **index of rigidity** of A, denoted ω_A (or ω when it is clear where it refers to) the number m/n^2, where m is the maximum number of zero entries for all the matrix power A^κ , as κ traces the naturals.

Definition 2.2: For two matrices A, B of the same size we will say that A is less rigid than B when $\omega_A < \omega_B$.

In case that $\omega_A = \omega_B$ we will say that And B have the same rigidit.

Remark 2.1: Evidently ω takes values in [0, 1) but in order to preserve compatibility to previous considerations we will set ω_A =1 whenever A is shrinkable.

Let us give now a few characteristic examples for various size matrices that involve numerical calculations in the quest of ω:

Example 2.1: Let A=$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which is the matrix representation of the 2-D DFT (see also Chapter 3).

Since A^2 =2I we evidently have periodically maximum sparsity 2/4 and thus we must take ω_A =1/2.

Example 2.2: Let A=$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Through the Cayley-Hamilton theorem, or after the calculation of A^2 directly,

we obtain A^2=A+I and thus ,recursively, if we set $A^\kappa = x_\kappa A + y_\kappa I$ we obtain that $x_{\kappa+1} = {}_\kappa x + x_{\kappa-1}$, for $\kappa \geq 2$ with $x_2 = x_1$ =1 and $y_\kappa = x_{\kappa-1}$,We conclude tha t $A^\kappa = F_\kappa A + F_{\kappa-1} I = \begin{pmatrix} F_\kappa + F_{\kappa-1} & F_\kappa \\ F_\kappa & F_{\kappa-1} \end{pmatrix}$, where

F_κ is the Fibonacci sequence. Thus ω_A =1 something that it was obvious to comclude in the first place but we gave the is proof since similar techniques and/or diagonalization will be needed in the case of matrices of more complicated numerical structure. It is also worth noticing that despite the "initial looks" the matrix of Example 2.1 is less rigid than the matrix of this example.

Example 2.3: Let A= $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.It is easy to check that spA={0,±√3 }.In order to study the sparsity of A^{κ}.

which is no longer evident like in the previous examples we can use Cayley-Hamilton theorem and apply iterations to obtain A^{κ} .starting from $A^3 = 3A$ .We have that $A^{3\rho} = 3^{\rho}A$, $A^{3\rho+1} = 3^{\rho}A^2$ and $A^{3\rho+2} = 3^{\rho+1}A$, for ρ=1,2,…Since it is easy to check that A^2 has no zero entries at all we infer that $\omega_A = 1/3$.Alternatively one could use the diagonalization $A^{\kappa}.= ED^{\kappa}E^{-1}$, with E the 3x3 matrix with columns the eigenvectors of the eigenvalues 0, √3 and -√3, respectively and $D^{\kappa} = \text{diag}(0, (\sqrt{3})^{\kappa}, (-\sqrt{3})^{\kappa})$ and then by splitting the result for κ=2ρ and κ=2ρ+1, ρ=1,2,…

This is clearly much more tedious method than the first but sometimes, when there is no reliable (from the rounding offs point of view) scientific program in hand, it is necessary in case the characteristic polynomial of A is in full form. We demonstrate thiw assertion in the example that follows.

Example 2.4: Let A= $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Its characteristic polynomial is $p(\lambda)=(2-\lambda)(\lambda^2-1)$ and the Cayley-Hamilton theorem leads to $A^3 = 2A^2+A-2I$ which produces tedious iterations in order to exqamine the sparsity of A^{κ}.

On the other hand it is easy to check that spA={-1,1,2}, that the eigenvectors

matrix is E= $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$ and that $E^{-1} = \frac{1}{6}\begin{pmatrix} 6 & -3 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{pmatrix}$. Since for κ=even and κ=odd__we obtain D^{κ} =diag(1,1,2^κ } and = diag(-1,1,2^κ }, respectively , we obtain finally that for κ=even

$A^{\kappa} = \frac{1}{6}\begin{pmatrix} 6 & 0 & \alpha_\kappa - 2 \\ 0 & 6 & 2\alpha_\kappa - 6 \\ 0 & 0 & 6\alpha_\kappa \end{pmatrix}$ and for κ=odd $A^{\kappa} = \frac{1}{6}\begin{pmatrix} -6 & 6 & \alpha_\kappa - 4 \\ 0 & 6 & 2\alpha_\kappa - 6 \\ 0 & 0 & 3\alpha_\kappa \end{pmatrix}$ with $\alpha_\kappa = 2^{\kappa+1}$.Thus $\omega_A = 1/2$.

Once again, since 1/2 >1/3, despite the "initial looks" we can say that our matrix is more rigid than the matrix of Example 2.3.The , meaning of such a- bizarre at first glance- remark is that although A has 4/9 sparsity one should not expect to increase sparsity in iterative algorithms that use matrix powers.

Example 2.5: Let A= $\begin{pmatrix} 1/3 & \alpha & 0 \\ 0 & 1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ where α is a random parameter. Evidently sp(A)={1/3, ½. 1}.

The diagonilazion A=Ediag(1/3,1/2.1}E^{-1}, with E= $\begin{pmatrix} 2 & 6\alpha & 0 \\ 0 & 1 & 0 \\ -3 & -12\alpha & 1 \end{pmatrix}$, E^{-1}= $\begin{pmatrix} 1/2 & -3\alpha & 0 \\ 0 & 1 & 0 \\ 3/2 & 3\alpha & 1 \end{pmatrix}$

leads to $A^{\kappa} = \begin{pmatrix} 1/3^{\kappa} & 6\alpha(1/2^{\kappa} - 1/3^{\kappa}) & 0 \\ 0 & 1/2^{\kappa} & 0 \\ \frac{3}{2}(1-1/3^{\kappa}) & 3\alpha(1+1/3^{\kappa-1} - 4/2^{\kappa}) & 1 \end{pmatrix}$. Thus for α≠0 and α=0, we can conclude that ω=4/9 and ω=5/9 respectively,

We also obtain $A^{\kappa} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.5 & 3\alpha & 1 \end{pmatrix}$, a .results useful in the norm estimations of A^{κ} (see Chapter 4).

Example 2.6: An interesting special class of matrices with applications in telecommunications and signal processing is that of **Hadamard** matrices (see e.g. [7]). A Hadamard matrix H_m is an m×m matrix with entries ±1 which satisfies the condition $H_m H_m^t$ =mI. By convention H_1 = (1) and the rest of the class, besides $H_2=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, consists of matrices of the order m=4^n, n=1,2,…

For example one such Hadamard matrix of the smallest size is $H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{pmatrix}$

(Note that it has been established for Hadamard matrices to write $-$ instead of -1).

In particular, symmetric Hadamard matrices H_m have a known spectrum (see e.g. [7]) which shows that their minimal polynomial is λ^2 -m. Then Cayley-Hamilton theorem leads to H_m^2 =mI and thus the index of rigidity of H_m is $\omega = \frac{m^2 - m}{m^2}$.

Remark 2.2: In many engineering applications, it is well known that the order, the complexity, the dimension of a model e.t.c can be expressed as the rank of a matrix Since evidently the more sparse a matrix is the smaller its rank, the minimizing of the rank in a sequence of matricial powere can facilitate calculations in System Theory(see e.g. the comments concerning MRP in [5].

Chapter 3. Sparsity and Rigidity of the nxn DFT Matrix

It is well known that the Discrete Fourier transform when performed on a time sample column nx1 to produce anx1 column of frequencies, can be equivalently described via the nxn matrix.

$$F_n = \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & z_n & z_n^{\ 2} & \dots & z_n^{\ n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & z_n^{\ n-2} & z_n^{\ 2(n-2)} & \dots & z_n^{\ (n-1)(n-2)} \\ 1 & z_n^{\ n-1} & z_n^{\ 2(n-1)} & \dots & z_n^{\ (n-1)^2} \end{pmatrix}, \text{ where } z_n = e^{2\pi i/n}\ , n\geq 2.$$

It is known that given any n the complete description of the n independent eigenvectors of F_n is still an open problem (see e.g. [8]). Even the question whether F_n is diagonalizable or not was settled rather recently Nevertheless ,without resorting to any diagonalization , with straightforward calculations, we will examine the index of rigidity and the sparsity of F_n^{κ} for every n and every κ.

We start with an evident lemma based on the definition of the n roots of unity:

Lemma: Let w be one of the n roots of unity with w≠1(and n≥2 to avoid trivialities). Then

(i) $\sum_{m=0}^{n-1} w^m$ =0. and from (i) we also get that

(ii) For n odd F_n^2 =nG, where G=(g_{ij}) is the nxn matrix with g_{11} =1and g_{ij} =0 for (i,j)≠(1,1).

(iii) For n odd F_n^3 =nH, where H=(h_{ij}) is the nxn matrix with h_{1j} =1and h_{ij} =0 for i≠1.

(iv) For n even F_n^2 =nI

Proposition: Lenna's (ii),(**iii**) & (iv) imply that

(i) For n odd , n>1 and ρ=1,2,…

(a) $F_n^{2\rho}$ $=n^{\rho}$ G, (b) $F_n^{2\rho+1} = n^{\rho}$ H

(ii) For n even.

(c) $F_n^{2\rho}$ $= n^{\rho}$ I, (d) $F_n^{2\rho+1} = n^{\rho} F_n$

Corollary: For n=odd and for n=even, the indices of rigidness of F_n are ω= 1/3 and ω=1/2 respectively.

Chapter 4. Sparsity and Rigidity in Matrix Norm Considerations

A matrix norm is a number defined in terms of the entries of the matrix. The norm is a useful quantity which can give important information about a matrix.

We give a concise account of the pertinent theory and limit ourselves to concepts and results tht directly or indirectly can be connected to the nature of a shrinkable and/or a rigid matrix.

(I) Matricial norms

The norm of a matrix is a measure of how large its elements are. The norm of a square matrix A is a non-negative real number denoted $\|A\|$. There are several different ways of defining a matrix norm, but they all share the following properties:

1. $\|A\| \geq 0$ for any square matrix A.

2. $\|A\| = 0$ if and only if the matrix A = 0.

3. $\|kA\| = .|k|\,\|A\|$, for any scalar k.

4. $\|A+B\| \leq \|A\| + \|B\|$

5. $\|AB\| \leq \|A\|\,\|B\|$

We restrict ourselves to nxn matrices A. The norm of a matrix is a measure of how large its elements are. It is a way of determining the "size" of a matrix that is not necessarily related to how big n is. In an abstract approach the norm of a nx1 column vector x induces the so called a natural norm of A through the "Operator Norm" formula $\|A\|$=sup $\|Ax\|/\|x\|$ for $\|x\| \neq 0$, which actually has the same value with max $\|Ax\|$ when $\|x\|$=1..Naturally the scalars ||x|| and ||Ax|| have to be computable at a "realistic" cost.

Remarks:

(a) All errors of th same norm are more or less equally significant or in significant.

(b) By definition, for any eigenvalue λ of A we have Av= λv for some eigenvector v ($\neq$o) and thus $|\lambda| \leq \|A\|$ |for every (compatible!) ||. ||, including every Operator Norm. Therefore a simple but very useful sufficient condition for our shrinkable matrices is the inequality ||A|| <1.

(c) In (II) we present the three more commonly used natural norms; actually there are so many that can be defined that emerges frequently the question how to chose an appropriate one .

(d) In (III) we will see some aspects of numerical analysis, mainly stemming from linear algebra techniques, where the concept of a matrix norm is heavily used; there it will be clear -especially when time and cost of computer use are key factors - possibly thelp one moght get examining in parallel the rigidity index and the potential for shrinkage of the matrices involved.

(e). Evidently property (I)5 leads to the inequality $\|A^n\| \leq \|A\|^n$ for all natural powers but, in general , this provides us with a very crude upper bound for $\|A^n\|$.especially when n is large and $\|A\|$>1.In (III) and even more so in Chapter 5 we will see how rigidity could be used in order to improve this upper bound.

(**II**) For a square matrix A, with real or complex entries we define now (**i**) the 1-norm**,** (**ii**) the infinity- norm, and (**iii**) the Euclidean norm, respectively, as follows:

(i) $\|A\|_1 = \max_j \sum_i |\alpha_{ij}|$ (also called the p=1 norm)

(ii) $\|A\|_\infty = \max_i \sum_j |\alpha_{ij}|$

(iii) $\|A\|_2 = [\sum\sum |\alpha ij|^2]^{1/2}$ (also called the p=2 norm)

Remarks:

(g) Though self evident it is useful to point out that the more sparse a matrix A the more easy the calculation of amy of the above norms; we will return to this state of affairs in (III) when $A=B^{\kappa}$ with B a shrinkable matrix or one with a very low index of rigidity.

(h) Another useful for our future considerations result of the theory is that all naturally inducrd matrix norms are equivalent in the sense that for any two such norms $\| \|_{\alpha}$ and $\| \|_{\beta}$ there exist positive constants c_1 and c_2 such that $c_1 \|A\|_{\alpha} \leq \|A\|_{\beta} \leq c_2 \|A\|_{\alpha}$ for every square matrix A.

(II) Matricial sequences and Limits

Example: Let $A=\begin{pmatrix} a & 1 & 0 \\ 0 & b & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and for simplicity set $\|A\|$ for $\|A\|_2$ and $(\mu_n)= \|A^n\|$.Using random matlab trials, under the restriction ab≠0 (in order to avoid trivialities), we obtain ,among others.:

Trial 1: For (a , b)= (1/3, 1/2) sequence (μ_n) is **increasing** and thus not any essential numerical advantage over $\|A^n\|_2 \leq (\|A\|_2)^n$ could be obtained.

On the other hand the next four trials produce a **decreasing** sequence (μ_n):

Trial 2: (a, b) = (-0.7,-0.2) with $\|A^{50}\| = \sqrt{1}, 586322338$

Trial 3: (a, b) =(-0.85,-0.38) with $\|A^{100}\| = \sqrt{1}, 445612382$

Trial 4: (a, b) = (-0.6, -0.4) with $\|A^{50}\| = \sqrt{1}, 589924745$

Trial 5: (a, b) = (-0.55,-0.5) with $\|A^{500}\| = \sqrt{1}, 601226038$

The most advantageous feature in the above four trials is that for n>50,100, 50 and 500, respectively when rounding off at the 9th decimal place the values 0f the termin each corresponding (μ_n) remains **constant**.

Chapter 5. A Sample of Certain Matricial Classifications

In this section, as a case study, with the assistance of matlab, we classify small size parametric families of matrices within the frame of all the theoretical concepts presented so far. We have avoided on purpose the large size matrix examples, but it is rather evident that when there is a reliable computer programming in hand we could classify all of them that are diagonazible along the same lines.

Example 5.1: We can directly check by hand that the following one-paramete familiies of 2x2 matrices are rigid under the given restructions

$$\begin{pmatrix} a & \frac{1-a}{2} \\ 1 & \frac{1}{2} \end{pmatrix} \text{ for } 1-\frac{1}{2} < a < \frac{3}{2}, a \neq 1 \text{ and } \begin{pmatrix} b & -2 \\ 1 & \frac{1}{b-1} \end{pmatrix} \text{ for } -\sqrt{3} < b < 1-\sqrt{2}.$$

Example 5.2: Let A=A(x,y)=$\begin{pmatrix} 0 & x \\ y & 1-xy \end{pmatrix}$, with the restriction the two real parameters x,y to be non zero and also xy≠1 (to avoid trivial cases. Directly by hand wecan check that A^2 has no zero entries and bt random matlab trials all A^n have no zero elements and also that there is a $n_0 = n_0$ (x,y) such that for n≥ n_0 A^n **does not change.** A heuristic result is also that the smaller or larger the value of $|x+y|$ the smaller or larger the value of n_0

Example 5.3: $\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$ No matter what the n is A^n will never excibit more than 5 zeros μηδενίζεται ενώ

Example 5.4: $\begin{pmatrix} \frac{3}{2} & -1 & 0 \\ -1 & \frac{3}{2} & -1 \\ 0 & -1 & \frac{3}{2} \end{pmatrix}$, A^2 has already no zeros and A^n “blows up” for n>664.

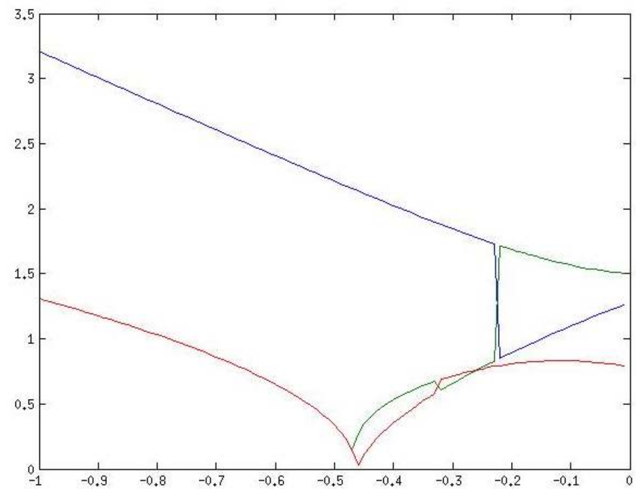
Example 5.5: We will firstly examine, in the frame of the previous discussion, whether is suitable to use in iterative powers algorithms the following one-parameter family of 4x4 real matrices:

A(α)=$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 1 & \alpha & -\alpha-\frac{1}{2} & 0 \\ 0 & -\alpha-\frac{1}{2} & \alpha & 1 \\ -\alpha-\frac{1}{2} & 0 & 1 & -\frac{1}{2} \end{pmatrix}$, where α≠$-\frac{1}{2}$ in order to avoid a “convenient “ 50% sparsity from starts. Since it is rather evident (by hand) that $-\frac{1}{2}$ is an eigenvalue of A(α) for all α, dividing the characteristic polynomial by α$+\frac{1}{2}$ we conclude that spA(α)={$-\frac{1}{2}$ ∪ S(α),
where S(α),is the set {λ: q(λ)=λ^3 +(1-3α)λ^2 + (2α^2 -4α -7/4)λ +4α^2 +3 α/4 -6/4 =0}.

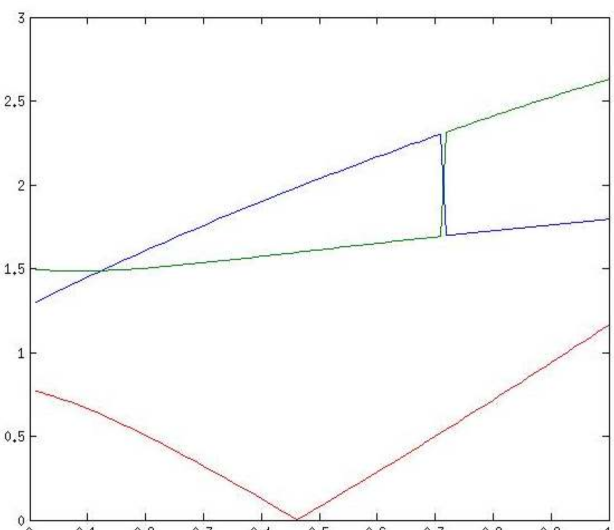
We are interested as α runs through the real numbers (α≠$-\frac{1}{2}$) to locate the number of the roots of q(λthat either they belong in (-1,1) or in the complex case to locate those that have a modulus <1.Note that on case we have, in addition to to the eigenvalue $-\frac{1}{2}$, two others in (-1,1) or two complex with a modulus

<1,then A^n is similar to D^n which for very large n is a good approximation , with respect to the $\| \|_2$ norm of the diagonal matrix diag (0,0,0,M)) where M =M(n) very large. We present now a rather representative of all cases sample from a large number of matlab trials:

- a = -0.794593 .Two roots (-0.879679 and -0.21315) are in (-1,1) while the third -2.29095 is not. There is **no increase** of zero entries for any A^n and actually it 'blows up" before A^{1100}
- **a = -0.835909.** Two roots (-0.788241 and -0.35905) are in (-1,1) while the third -2.36044,ps not. There is no increase of zero entries for any A^n and actually it 'blows up' before A^{1100}
- **a = -0.916835.**Two (conjugate) complex roots 0.626076 + 0.279692i with modulus<1 but unfortunately the third one -2.49835 is not in (-1,1) and we have similar results aw before
- **a = 0.5 .**Only one root = -0.0387099 is in (-1,1),while the two others (-1.547 and 2.08641) are not and of course once again we have similar results as before



Example 5.5. Graph 1



Example 5.5. Graph 2

Chapter 6. Conclusions

It is well known that when a (square) matrix is modeling a linear problem (like e.g. solving a linear system) and especially in cases where an iterative powers algorithm will be involved (see e.g. [9,10] or [11]), there are some special numerical features like the spectrum, the column and/or row norm, the determinant the system's state index, the sparsity and a few others that can be used in order to classify the matrix and/or predict if the algorithm is a good one from the aspect of cost in time , robustness the degree of complexity of the algebraic operations that will be necessary e.t.c.. Im our examples we have shown (restricting our selves mostly to small size matrices for simplicity) tha in some cases the initial matrix could be looking as a " promising one" but soon or in the long run becomes less and less usable; while, on the other hand some looking less promising (e.g. with a small or no spasrsity at all) turn out to be more usable without eventually the need of any remodeling and any pertinent re-scaling of the initial matrix m the sense of equilibration of data in the linear systems of equations. Thus, the new numerical feature that we introduced , namely the index of rigidity , is propose to be included among the other numbers, for matrix classification textbooks (e.g. [12] or [13]) or online manuals (like e.g. KB. Petersen & M S Pedersen's, Matrix Cookbook, version November 2012).

Competing Interests

Authors have declared that no competing interests exist.

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