British Journal of Mathematics & Computer Science
13(2): 1-20, 2016, Article no.BJMCS.21903
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

The Complementary Exponentiated Inverted Weibull Power Series Family of Distributions and its Applications

Amal S. Hassan[1], A. M. Abd-Elfattah[1] and Asmaa H. Mokhtar[1*]

[1]*Institute of Statistical Studies and Research, Cairo University, Egypt.*

Article Information

DOI: 10.9734/BJMCS/2016/21903
Editor(s):
(1) H. M. Srivastava, Department of Mathematics and Statistics, University of Victoria, Canada.
Reviewers:
(1) G. Y. Sheu, Chang-Jung Christian University, Tainan, Taiwan.
(2) P. E. Oguntunde, Covenant University, Nigeria.
(3) Thomas L. Toulias, Technological Educational Institute of Athens, Greece.
Complete Peer review History: http://sciencedomain.org/review-history/12518

Original Research Article

Received: 08 September 2015
Accepted: 03 November 2015
Published: 02 December 2015

Abstract

In statistical literature, several distributions are used to model lifetime data. But many –if not most– of these distributions lack of certain lifetime context motivation. Recently, attempts have been made to define new families of probability distributions that extended well known families of distributions and at the same time offer great flexibility in a real data modeling. In this paper, a new family of distributions is introduced by compounding exponentiated inverted Weibull and power series distributions. The properties of the new family are discussed, including quantiles, moments and moment generating function. The estimation of the model parameters is performed by the maximum likelihood method. The new family generalizes some new lifetime distributions. In particular, two special models belonging to this family are studied in some details. Applications to real data sets are given to show the flexibility and potentiality of the proposed family of distributions.

Keywords: Exponentiated inverted Weibull distribution; power series distribution; order statistics; hazard function.

1 Introduction

In recent years, compound distributions arise and applied in several areas, such as public health, economics, engineering, and industrial reliability. New several compound models of distributions have been introduced follows the same way that was previously carried out in [1]. Many researchers have proposed a series of new compounding distributions by mixing the distribution when the lifetime can be expressed as the minimum or maximum of a sequence of independent and identically distributed, random variables which represent the failure time of a system.

**Corresponding author: E-mail: asmaa_hussein2005@yahoo.com;*

Several authors have proposed new compounding distributions by mixing the distribution of a maximum of a fixed number of independent for any continuous lifetime distributed random variables and discrete random variable. The exponential truncated Poisson with increasing failure rate has been considered in [2]. A new lifetime distribution, which is called the exponentiated Weibull-geometric distribution, was introduced in [3]. This new distribution was obtained by compounding the exponentiated Weibull together with geometric distributions. For more about some other compounding distributions; can be found in [4-14].

Furthermore, several authors derived new compounding distributions by mixing continuous distributions with power series distribution for both schemes (minimum or maximum). In [15] the Weibull power series family of distributions, which includes the sub-models of the exponential power series distributions, was defined. The generalized exponential power series distribution was introduced in [16]. A new class of extended Weibull power series distributions was introduced in [17]. Two compound families, namely the max-Erlang power series distribution and the min-Erlang power series distribution were proposed in [18]. A new family of Burr XII power series models was introduced in [19]. A new class of lifetime distributions called the Lindley power series was proposed in [20]. A new life time distribution with decreasing failure rate, called the inverse Weibull power series distribution was, introduced in [21].

In this article, a new lifetime family of distributions is introduced by compounding exponentiated inverted Weibull and power series distributions. The density, cumulative, survival and hazard rate functions of the new family are introduced in Section 2. In Section 3, some mathematical properties of the new family are obtained such as, quantiles and moments. In the same section, some distributions of order statistics are derived. In Section 4, some special sub-models and mathematical properties of the complementary exponentiated inverted Weibull Poisson and the complementary exponentiated inverted Weibull logarithmic distributions are given. In Section 5, maximum likelihood estimator of the unknown parameters for the family is obtained. In Section 6, three illustrative examples based on real data set are provided. Finally, the concluding remarks are addressed in Section 7.

2 The New Family of Distributions

The inverse Weibull distribution is one of the most popular probability distributions to model life time data with some monotone failure rates. The usefulness and applications of the inverse Weibull distribution in various areas including reliability and branching processes, can be seen in [22] and in references therein. Exponentiated (generalized) inverted Weibull distribution is a generalization to the inverted Weibull distribution through adding a new shape parameter. The probability density function (pdf) of exponentiated inverted Weibull (EIW) with two shape parameters; takes the following form

$$g(y;\theta,\beta)=\theta\beta\, y^{-(\beta+1)}e^{-\theta y^{-\beta}}, \qquad y,\theta,\beta>0. \tag{1}$$

The corresponding cumulative distribution function (cdf) is given by

$$G(y;\theta,\beta)=e^{-\theta y^{-\beta}}, \qquad y,\theta,\beta>0. \tag{2}$$

For $\theta=1$, it represents the standard inverted Weibull distribution, for $\beta=1$, it represents the exponentiated standard inverted exponential distribution as mentioned in [23].

Let $Y=\max\{Y_i\}_{i=1}^{Z}$, be independent and identically distributed random variables following the exponentiated inverted Weibull distribution whose density function is given by (1) with shape parameters $\theta>0$ and $\beta>0$, while Z is a discrete random variable following the power series with probability mass function (truncated at zero).

$$\mathrm{P}(Z=z;\,\lambda)=\frac{a_z\,\lambda^z}{A(\lambda)}, \qquad z=1,2,\ldots \tag{3}$$

The coefficient a_z's depends only on z , $A(\lambda) \coloneqq \sum_{z=1}^{\infty} a_z \lambda^z,\ \ \lambda > 0,$ is such that $A(\infty)$ is finite.

Table 1. Useful quantities of some Power series distributions

Distributions	a_z	$A(\lambda)$	$A'(\lambda)$	$A''(\lambda)$	$A(\lambda)^{-1}$	λ
Poisson	$z!^{-1}$	$e^{\lambda}-1$	e^{λ}	e^{λ}	$\ln(\lambda+1)$	$\lambda \in (0,\infty)$
Logarithm	z^{-1}	$-\ln(1-\lambda)$	$(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$1-e^{-\lambda}$	$\lambda \in (0,1)$
Geometric	1	$\lambda(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$2(1-\lambda)^{-3}$	$\lambda(\lambda+1)^{-1}$	$\lambda \in (0,1)$
Binomial	$\binom{m}{z}$	$(\lambda+1)^m-1$	$m(\lambda+1)^{m-1}$	$\frac{m(m-1)}{(\lambda+1)^{2-m}}$	$(\lambda-1)^{\frac{1}{m}}-1$	$\lambda \in (0,\infty)$

In [6], a compound class of distributions was introduced. These distributions were obtained by mixing the maximum of a sequence of any identically independent lifetime distributed random variables together with a power series random variable. The cumulative distribution function of this class is given by

$$F\left(y;\lambda,\underline{\phi}\right)=\frac{A\left(\lambda\, G\left(y,\underline{\phi}\,\right)\right)}{A(\lambda)} \qquad ,y>0,\quad \theta,\beta,\lambda>0, \qquad \underline{\phi}=(\phi_1,\phi_2,\dots,\phi_l), \tag{4}$$

where $G(y)$ is the cdf of any lifetime distribution. Therefore, the distribution function of the complementary exponentiated inverted Weibull power series (CEIWPS) family of distributions is obtained by substituting cdf (2) in cdf (4) as follows

$$F(y;\theta,\beta,\lambda)=\frac{A\left(\lambda\, e^{-\theta y^{-\beta}}\right)}{A(\lambda)}\,, \qquad y>0,\quad \theta,\beta,\lambda>0. \tag{5}$$

The pdf corresponding to (5) is given by,

$$f(y;\theta,\beta,\lambda)=\frac{\lambda\theta\beta\, y^{-(\beta+1)}e^{-\theta y^{-\beta}}\, A'\left(\lambda\, e^{-\theta y^{-\beta}}\right)}{A(\lambda)} \qquad ,y>0,\quad \theta,\beta,\lambda>0. \tag{6}$$

A random variable Y distributed as in (5) shall be denoted by $Y \sim$ CEIWPS $\left(\underline{\psi}\right)$, where $\underline{\psi} \equiv (\theta,\beta,\lambda)$ is an unknown vector of parameters. Furthermore, the reliability and hazard rate functions take the following forms

$$R\left(y;\underline{\psi}\right)=1-\frac{A\left(\lambda\, e^{-\theta y^{-\beta}}\right)}{A(\lambda)} \qquad ,y>0, \qquad \theta,\beta,\lambda>0. \tag{7}$$

and

$$h\left(y;\underline{\psi}\right)=\frac{\lambda\theta\beta\, y^{-(\beta+1)}e^{-\theta y^{-\beta}}\, A'\left(\lambda\, e^{-\theta y^{-\beta}}\right)}{A(\lambda)-A\left(\lambda\, e^{-\theta y^{-\beta}}\right)} \qquad ,y>0, \qquad \theta,\beta,\lambda>0. \tag{8}$$

3 Properties of the Family

In this section, some statistical properties of the new family, including quantiles, moments and moment generating function are obtained. In addition, some distributions of order statistics are studied.

3.1 A Useful Expansion

Some properties of the cdf (5) and pdf (6) will be studied through the following two Propositions.

Proposition 1

The EIW distribution is a limiting special case of EIWPS family of distributions when $\lambda \to 0.$

Proof

By substituting $A(\lambda) \coloneqq \sum_{z=1}^{\infty} a_z \lambda^z,\ \ \lambda > 0,$ in cdf (5), then it can be written as

$$F(y;\theta,\beta,\lambda) = \frac{\sum_{z=1}^{\infty} a_z \left(\lambda e^{-\theta y^{-\beta}}\right)^z}{\sum_{z=1}^{\infty} a_z \lambda^z},$$

by taking the limits for both sides,

$$\underset{\lambda \to 0}{Lim} F(y;\theta,\beta,\lambda) = \frac{a_1 e^{-\theta y^{-\beta}} + \sum_{z=1}^{\infty} z\, a_z \left(\lambda e^{-\theta y^{-\beta}}\right)^{z-1} e^{-\theta y^{-\beta}}}{a_1 + \sum_{z=1}^{\infty} z\, a_z \lambda^{z-1}} = e^{-\theta y^{-\beta}},$$

which is the cdf of EIW distribution.

Proposition 2

The pdf of the complementary exponentiated inverted Weibull power series family of distributions can be expressed as an infinite mixture of exponentiated inverted Weibull with parameters $(\theta z, \beta)$ as follows

$$f(y;\theta,\beta,\lambda) = \sum_{z=1}^{\infty} P(Z = z;\lambda) g(y;\ \theta z, \beta),$$

where $g(y)$ is the pdf of $Y \sim EIW(\theta z, \beta)$.

Proof

Substituting $A'(\lambda) \coloneqq \sum_{z=1}^{\infty} z\, a_z \lambda^{z-1},\ \ \lambda > 0,$ in pdf (6), then we get

$$f(y;\theta,\beta,\lambda) = \frac{\lambda\theta\beta}{A(\lambda)}\, y^{-(\beta+1)} e^{-\theta y^{-\beta}} \sum_{z=1}^{\infty} z\, a_z \ \left(\lambda e^{-\theta y^{-\beta}}\right)^{z-1},$$

$$f(y;\theta,\beta,\lambda) = \frac{\sum_{i=1}^{z} a_z \lambda^z\, z\,\theta\,\beta\, y^{-(\beta+1)} \left(e^{-\theta z y^{-\beta}}\right)^z}{A(\lambda)},$$

$$f(y;\theta,\beta,\lambda) = \sum_{z=1}^{\infty} P(Z = z) g(y;\ \theta z, \beta).$$

3.2 Quantiles and Moments

Quantile functions are used in theoretical aspects of probability theory, statistical applications and simulations. Simulation methods utilize quantile function to produce simulated random variables for classical and new continuous distributions. The quantile function, say $Q(u)$ of Y is given by

$$u=\frac{A\left(\lambda\, e^{-\theta Q(u)^{-\beta}}\right)}{A(\lambda)},$$

after some simplifications, it reduces to the following form

$$Q(u)=\left[-\frac{1}{\theta}\ln\left\{\frac{A^{-1}(A(\lambda)u)}{\lambda}\right\}\right]^{\frac{-1}{\beta}}. \tag{9}$$

Where, u is considered as a uniform random variable on the unit interval $(0,1)$ and $A^{-1}(.)$ is the inverse function of $A(.)$. The r-th moment of Y about the origin follows from proportion 2 as follows

$$\acute{\mu}_r=E(Y^r)=\int_0^{\infty}\sum_{z=1}^{\infty}P(Z=z)g(y;\,\theta z,\beta)\;\; y^{r-(\beta+1)}\;dy.$$

According to [23], the *r*-th moment for random variable Y', which follows an EIW distribution with parameters θ and β is given by $\theta^{\frac{r}{\beta}}\,\Gamma\left(1-\frac{r}{\beta}\right)$. It is easy to show that, by using this property as mentioned in [23], the *r*-th moment of Y following the CEIWPS distribution with parameters $(\theta z,\beta)$ is given by

$$\acute{\mu}_r=\sum_{z=1}^{\infty}P(Z=z)\;(\theta z)^{\frac{r}{\beta}}\,\Gamma\left(1-\frac{r}{\beta}\right),\qquad \mathrm{r}=1,2,3,\ldots \tag{10}$$

Also, it is easy to show that,

$$M_Y(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\,\acute{\mu}_r,$$

where, $\acute{\mu}_r$ is the *r*-th moment**,** while $M_Y(t)$ denotes the moment generating function (mgf) of Y. Then by using (10), the mgf of Y can be written as

$$M_Y(t)=\sum_{r=0}^{\infty}\sum_{z=1}^{\infty}\frac{t^r}{r!}P(Z=z)(\theta z)^{\frac{r}{\beta}}\,\Gamma\left(1-\frac{r}{\beta}\right).$$

3.3 Some Distributions of Order Statistics

Let $Y_{1:n}<Y_{2:n}<\ldots<Y_{n:n}$ be the order statistics of a random sample of size n following the exponentiated inverted Weibull power series family of distributions, with parameters $\theta,\beta,$ and $\lambda,$ then as mentioned in [24], the pdf of the *i*-th order statistic, can be written as follows

$$f_{i:n}\left(y;\underline{\psi}\right)=\frac{f\left(y;\underline{\psi}\right)}{B(i,n-i+1)}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\left[F\left(y;\underline{\psi}\right)\right]^{i+j-1}\qquad ,0<y<\infty,$$

$$f_{i:n}\left(y;\underline{\psi}\right)=\frac{\lambda\theta\beta\, y^{-(\beta+1)}e^{-\theta y^{-\beta}}A'\left(\lambda\, e^{-\theta y^{-\beta}}\right)}{\mathrm{B}(i,n-i+1)[\,A(\lambda)]^{i+j}}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\left[A\left(\lambda\, e^{-\theta y^{-\beta}}\right)\right]^{i+j-1}, \qquad (11)$$

where $\mathrm{B}(.,.)$ denotes the beta function. The expansion for $A\left(\lambda\, e^{-\theta y^{-\beta}}\right)^{i+j-1}$ is given by

$$\begin{aligned} A\left(\lambda\; e^{-\theta y^{-\beta}}\right)^{i+j-1} &= \left[\sum_{z=1}^{\infty} a_z\;\; \lambda^z \left(e^{-\theta y^{-\beta}}\right)^z\right]^{i+j-1}, \\ &= \left[\, a_1\lambda e^{-\theta y^{-\beta}}\left[1+\frac{a_2}{a_1}\lambda\, e^{-\theta y^{-\beta}}+\frac{a_3}{a_1}\lambda^2\left(e^{-\theta y^{-\beta}}\right)^2+\cdots\right]\right]^{i+j-1}, \\ &= a_1^{i+j-1}\lambda^{i+j-1}\left(e^{-\theta y^{-\beta}}\right)^{(i+j-1)}\left[\sum_{m=0}^{\infty} b_m\,\lambda^m\left(e^{-\theta y^{-\beta}}\right)^m\right]^{i+j-1}, \end{aligned}$$

where $b_m=\frac{a_{m+1}}{a_1}$ for $m=1,2,3,\ldots$

By using the identity $(\sum_{m=0}^{\infty} b_m\, z^m)^j \coloneqq \sum_{m=0}^{\infty} d_{j:m} z^m$ for positive integer j as mentioned in [25]), then

$$A\left(\lambda\, e^{-\theta y^{-\beta}}\right)^{i+j-1}=\sum_{m=0}^{\infty} d_{i+j-1:m}\;\, a_1^{i+j-1}\lambda^{i+j+m-1}\left(\, e^{-\theta y^{-\beta}}\right)^{(i+j+m-1)}. \qquad (12)$$

By using the expansion for $A'(\lambda) \coloneqq \sum_{z=1}^{\infty} z\; a_z \lambda^{z-1},\;\; \lambda>0$, then

$$A'\left(\lambda\, e^{-\theta y_{(n)}{}^{-\beta}}\right)=\sum_{k=1}^{\infty} k\; a_k\; \lambda^{k-1}\left(e^{-\theta y^{-\beta}}\right)^{(k-1)}=a_1\sum_{k=0}^{\infty}(k+1)\, b_k\lambda^k\;\left(e^{-\theta y^{-\beta}}\right)^k, \qquad (13)$$

substituting (12) and (13) in (11), we get

$$f_{i:n}\left(y;\underline{\psi}\right)=\frac{\theta\beta\; y^{-(\beta+1)}}{B(i,n-i+1)A(\lambda)^{j+i}}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{n-i}{j}(-1)^j\;\, b_k(k+1)$$

$$\times\, d_{i+j-1:m}\;\; a_1^{i+j}\lambda^{i+j+m+k}\left(e^{-\theta y^{-\beta}}\right)^{(i+j+m+k)},$$

$$f_{i:n}\left(y\,;\underline{\psi}\right)=\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\omega_{i,j,k,m}\quad y^{-(\beta+1)}\;\left(e^{-\theta y^{-\beta}}\right)^{(i+j+m+k)}, \qquad (14)$$

where,

$$\omega_{i,j,k,m}\!:=\frac{\theta\beta\binom{n-i}{j}(-1)^j\, b_k\,(k+1)\, d_{i+j-1:m}\quad a_1^{i+j}\lambda^{i+j+m+k}}{B(i,n-i+1)A(\lambda)^{j+i}}.$$

The pdf of the smallest order statistic is given by substituting $i=1$ into (14) as follows

$$f_{1:n}\left(y;\underline{\psi}\right)=\sum_{j=0}^{n-1}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\omega_{1,j,k,m}\quad y^{-(\beta+1)}\left(e^{-\theta y^{-\beta}}\right)^{(j+m+k+1)}\;\; ,0<y<\infty,$$

where

$$\omega_{1,j,k,m} = \frac{n\,\theta\beta\binom{n-1}{j}(-1)^j b_k(k+1)\,d_{j:m} \quad a_1^{j+1}\lambda^{j+m+k+1}}{A(\lambda)^{j+1}}.$$

The pdf of the largest order statistic is obtained by substituting $i = n$ into (14) as follows

$$f_{n:n}\left(y;\underline{\psi}\right) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\omega_{n,j,k,m} \quad y^{-(\beta+1)}\left(e^{-\theta y^{-\beta}}\right)^{(n+j+m+k)} \quad ,0 < y < \infty,$$

where

$$\omega_{n,j,k,m} = \frac{n\,\theta\beta(-1)^j b_k(k+1)\,d_{n+j-1:m} \quad a_1^{n+j}\lambda^{n+j+m+k}}{A(\lambda)^{j+n}}.$$

4 Special Models of the Family

In this section, some special sub-models of the complementary exponentiated inverted power series family of distributions are studied. The complementary exponentiated inverted Weibull Poisson (CEIWP) and the complementary exponentiated inverted Weibull logarithmic (CEIWL) distributions are discussed in some details.

The sub-models are considered as follows:

1- For $A(\lambda) := e^{\lambda} - 1$, the CEIWPS family of distributions reduces to the CEIWP distribution with the following cdf

$$F(y;\theta,\beta,\lambda) = \frac{\exp\left(\lambda\, e^{-\theta y^{-\beta}}\right) - 1}{(e^{\lambda} - 1)}, \qquad y,\theta,\beta,\lambda > 0. \tag{15}$$

2- For $A(\lambda) := -\ln(1-\lambda)$, the CEIWPS family of distributions reduces to the CEIWL distribution with the following cdf

$$F(y;\theta,\beta,\lambda) = \frac{-\ln\left(1 - \lambda\, e^{-\theta y^{-\beta}}\right)}{-\ln(1-\lambda)}, \quad y,\theta,\beta > 0, \qquad 0 < \lambda < 1. \tag{16}$$

3- For $(\lambda) := \lambda(1-\lambda)^{-1}$, the CEIWPS family of distributions reduces to the complementary exponentiated inverted Weibull geometric distribution with the following cdf

$$F(y;\theta,\beta,\lambda) = \frac{(1-\lambda)e^{-\theta y^{-\beta}}}{\left(1-\lambda e^{-\theta y^{-\beta}}\right)}, \qquad y,\theta,\beta > 0. \qquad 0 < \lambda < 1.$$

4- For $(\lambda) := (\lambda+1)^m - 1$, the CEIWPS family of distributions reduces to the complementary exponentiated inverted Weibull binomial distribution with the following cdf

$$F(y;\theta,\beta,\lambda) = \frac{\left(\lambda e^{-\theta y^{-\beta}} + 1\right)^m - 1}{(\lambda+1)^m - 1}, \qquad y,\theta,\beta,\lambda > 0.$$

5- For $\theta = 1$, $A(\lambda) := e^{\lambda} - 1$, the CEIWPS family of distributions reduces to the complementary standard inverted Weibull Poisson distribution with the following cdf

$$F(y;\beta,\lambda) = \frac{\exp\left(\lambda\, e^{-y^{-\beta}}\right) - 1}{(e^{\lambda} - 1)}, \qquad y, \beta, \lambda > 0 .$$

6- For $\theta = 1$, $A(\lambda) := -\ln(1-\lambda)$, the CEIWPS family of distributions reduces to the complementary standard inverted Weibull logarithmic distribution with the following cdf

$$F(y;\beta,\lambda) = \frac{-\ln\left(1 - \lambda\, e^{-y^{-\beta}}\right)}{-\ln(1-\lambda)}, \qquad y, \beta > 0, \qquad 0 < \lambda < 1.$$

7- For $\theta = 1$, $A(\lambda) := \lambda(1-\lambda)^{-1}$, the CEIWPS family of distributions reduces to the complementary standard inverted Weibull geometric distribution with the following cdf

$$F(y;\beta,\lambda) = \frac{(1-\lambda)\, e^{-y^{-\beta}}}{\left(1 - \lambda\, e^{-y^{-\beta}}\right)}, \qquad y, \beta > 0. \qquad 0 < \lambda < 1.$$

8- For $\theta = 1$, $A(\lambda) := (1+\lambda)^m - 1$, the CEIWPS family of distributions reduces to the complementary standard inverted Weibull binomial distribution with the following cdf

$$F(y;\beta,\lambda) = \frac{\left(\lambda\, e^{-y^{-\beta}} + 1\right)^m - 1}{(\lambda+1)^m - 1}, \quad y, \beta, \lambda > 0.$$

9- For $\beta = 1$, $A(\lambda) := e^{\lambda} - 1$, the CEIWPS family of distributions reduces to the complementary exponentiated standard inverted exponential Poisson distribution with the following cdf

$$F(y;\theta,\lambda) = \frac{\exp\left(\lambda\, e^{\frac{-\theta}{y}}\right) - 1}{(e^{\lambda} - 1)}, \quad y, \theta, \lambda > 0.$$

10- For $\beta = 1$, $A(\lambda) := -\ln(1-\lambda)$, the CEIWPS family of distributions reduces to the complementary exponentiated standard inverted exponential logarithmic distribution with the following cdf

$$F(y;\theta,\lambda) = \frac{-\ln\left(1 - \lambda\, e^{\frac{-\theta}{y}}\right)}{-\ln(1-\lambda)}, \qquad y, \theta > 0, \qquad 0 < \lambda < 1.$$

11- For $\beta = 1$, $A(\lambda) := (1-\lambda)^{-1}$, the CEIWPS family of distributions reduces to the complementary exponentiated standard inverted exponential geometric distribution with the following cdf

$$F(y;\theta,\lambda) = \frac{e^{\frac{-\theta}{y}}\,(1-\lambda)}{\left(1 - \lambda\, e^{\frac{-\theta}{y}}\right)}, \qquad y, \theta > 0, \qquad 0 < \lambda < 1.$$

12- For $\beta = 1$, $A(\lambda) := (1+\lambda)^m - 1$, the CEIWPS family of distributions reduces to the complementary exponentiated standard inverted exponential binomial distribution with the following cdf

$$F(y;\theta,\lambda) = \frac{\left(\lambda\, e^{\frac{-\theta}{y}} + 1\right)^m - 1}{(\lambda+1)^m - 1}, \qquad y, \theta, \lambda > 0.$$

4.1 The Complementary Exponentiated Inverted Weibull Poisson Distribution

In this subsection, the pdf, reliability, hazard function, quantile and moments for the complementary exponentiated inverted Weibull Poisson distribution are studied in more details. The pdf of the CEIWP corresponding to (15) is as follows

$$f(y;\theta,\beta,\lambda) = \frac{\theta\beta\,\lambda}{(e^{\lambda} - 1)}\, y^{-(\beta+1)}\, e^{-y^{-\theta\beta}} \exp\left(\lambda e^{-\theta y^{-\beta}}\right), \qquad y, \theta, \beta, \lambda > 0. \tag{17}$$

Figs. 1 and 2 represent the pdf and the cdf of the CEIWP distribution for some selected values of parameters θ, β and λ .

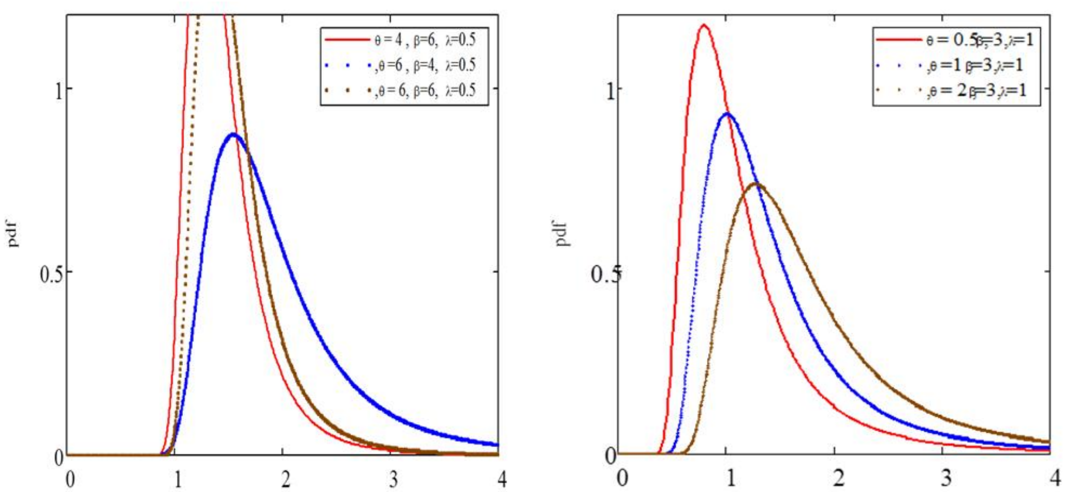

Fig. 1. Plots of the CEIWP densities for some parameter values

The reliability and hazard rate functions of the CEIWP are obtained respectively, as follows

$$R(y;\theta,\beta,\lambda) = \frac{e^{\lambda} - \exp\left(\lambda e^{-\theta y^{-\beta}}\right)}{(e^{\lambda} - 1)}, \qquad y, \theta, \beta, \lambda > 0, \tag{18}$$

and,

$$h(y;\,\theta,\beta,\lambda) = \frac{\theta\beta\lambda\;y^{-(\beta+1)}\, e^{-\theta y^{-\beta}} \exp\left(\lambda e^{-\theta y^{-\beta}}\right)}{e^{\lambda} - \exp\left(\lambda\, e^{-\theta y^{-\beta}}\right)}, \qquad y, \theta, \beta, \lambda > 0. \tag{19}$$

Figs. 3 and 4 illustrate the reliability and the hazard rate functions of the CEIWP distribution for the same set of selected parameters as in Figs. 1 and 2.

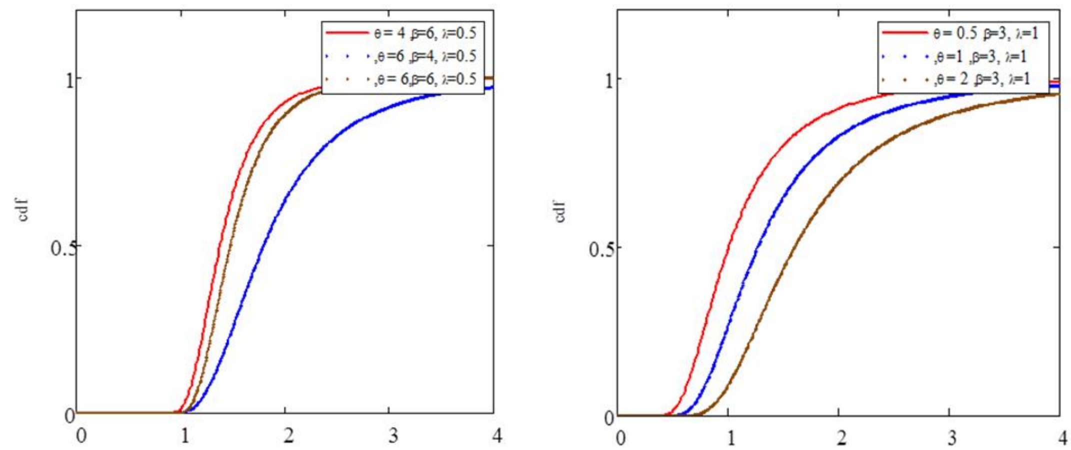

Fig. 2. Plots of the CEIWP distribution function for some parameter values

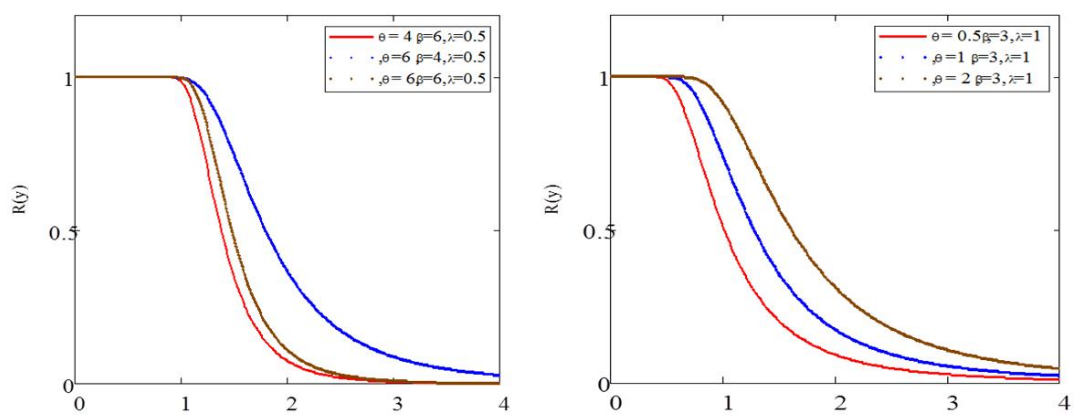

Fig. 3. Plots of the CEIWP reliability functions for some parameter values

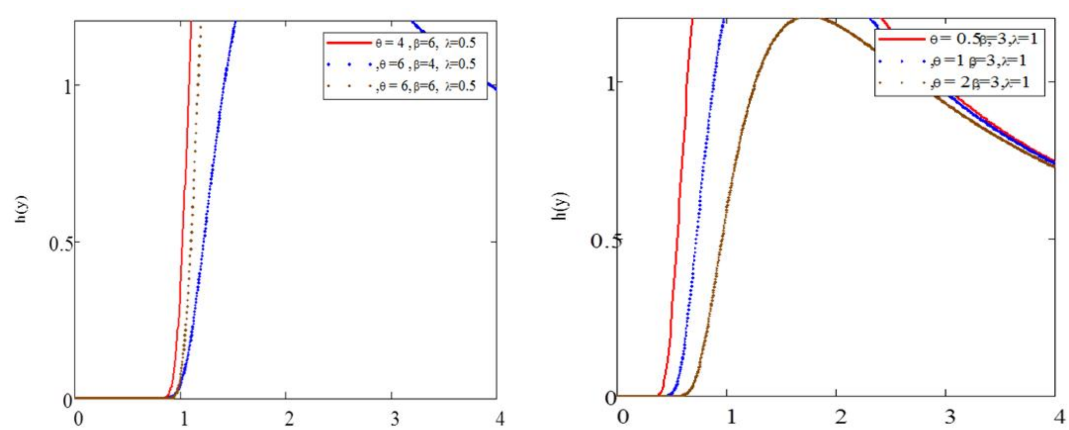

Fig. 4. Plots of the CEIWP hazard rates functions for some parameter values

It is clear from Fig. 4 that the hazard function takes different forms according to different values of the parameters.

The quantile function of the CEIWP distribution is given by substituting $A(\lambda) := e^{\lambda} - 1$, and $A^{-1}(\lambda) \coloneqq \ln(\lambda + 1)$ in (9), as follows

$$Q(u) = \left[-\frac{1}{\theta}\ln\left(\frac{1}{\lambda}\ln\{u(e^{\lambda} - 1) + 1\}\right)\right]^{\frac{-1}{\beta}}, \tag{20}$$

where u is a uniform random variable on the unit interval (0,1). In particular, the median of the CEIWP distribution, say Q_2, is obtained by setting $u := \frac{1}{2}$ into (20), i.e.

$$Q_2 = \left[-\frac{1}{\theta}\ln\left(\frac{1}{\lambda}\ln\frac{e^{\lambda} + 1}{2}\right)\right]^{\frac{-1}{\beta}}, \qquad \lambda > 0. \tag{21}$$

The *r*-th moment of the CEIWP distribution about the origin is given by setting $A(\lambda) := e^{\lambda} - 1$ into (10), i.e.

$$\acute{\mu}_r = \sum_{z=1}^{\infty} \frac{a_z \lambda^z}{(e^{\lambda} - 1)} (\theta z)^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right), \qquad r = 1,2,3, \dots \;. \tag{22}$$

In particular, setting $r = 1$ in (22), the mean of the CEIWP distribution is given by

$$\mathrm{E}(Y) = \sum_{z=1}^{\infty} \frac{a_z \lambda^z}{(e^{\lambda} - 1)} (\theta z)^{\frac{1}{\beta}} \Gamma\left(1 - \frac{1}{\beta}\right). \tag{23}$$

The variance of the CEIWP distribution takes the following form

$$\mathrm{Var}(Y) = \sum_{z=1}^{\infty} \frac{a_z \lambda^z}{(e^{\lambda} - 1)} (\theta z)^{\frac{2}{\beta}} \Gamma\left(1 - \frac{2}{\beta}\right) - \left[\sum_{z=1}^{\infty} \frac{a_z \lambda^z}{(e^{\lambda} - 1)} (\theta z)^{\frac{1}{\beta}} \Gamma\left(1 - \frac{1}{\beta}\right)\right]^2.$$

4.2 The Complementary Exponentiated Inverted Weibull logarithmic Distribution

In this subsection, the pdf, reliability, hazard function, quantile, and moments for the complementary exponentiated inverted Weibull logarithmic distribution are discussed.

The probability density function of the CEIWL distribution corresponding to (16) is given by

$$f(y;\theta,\beta,\lambda) = \frac{\theta\beta\,\lambda}{-\ln(1-\lambda)\left(1-\lambda e^{-\theta y^{-\beta}}\right)}\, y^{-(\beta+1)}\, e^{-\theta\, y^{-\beta}}, \quad y, \theta, \beta, \lambda > 0. \tag{24}$$

Figs. 5 and 6 represent the pdf and the cdf of the CEIWL distribution for selected values of parameters θ, β and λ.

The reliability and hazard rate functions of the CEIWL distribution are obtained respectively, as follows

$$R(y;\theta,\beta,\lambda) = 1 - \frac{\ln\left(1 - \lambda\, e^{-\theta y^{-\beta}}\right)}{\ln(1-\lambda)}, \qquad y, \theta, \beta, \lambda > 0, \tag{25}$$

and,

$$h(y;\theta,\beta,\lambda\,) = \frac{\theta\beta\lambda\ y^{-(\beta+1)}\, e^{-\theta y^{-\beta}}\left(1-\lambda\, e^{-\theta y^{-\beta}}\right)^{-1}}{-\ \ln(1-\lambda)+\ln\left(1-\lambda\, e^{-\theta y^{-\beta}}\right)}, \qquad y,\theta,\beta,\lambda > 0. \tag{26}$$

Figs. 7 and 8 illustrate the reliability and hazard rate functions of the CEIWL for the same set of parameters as in Figs. 5 and 6.

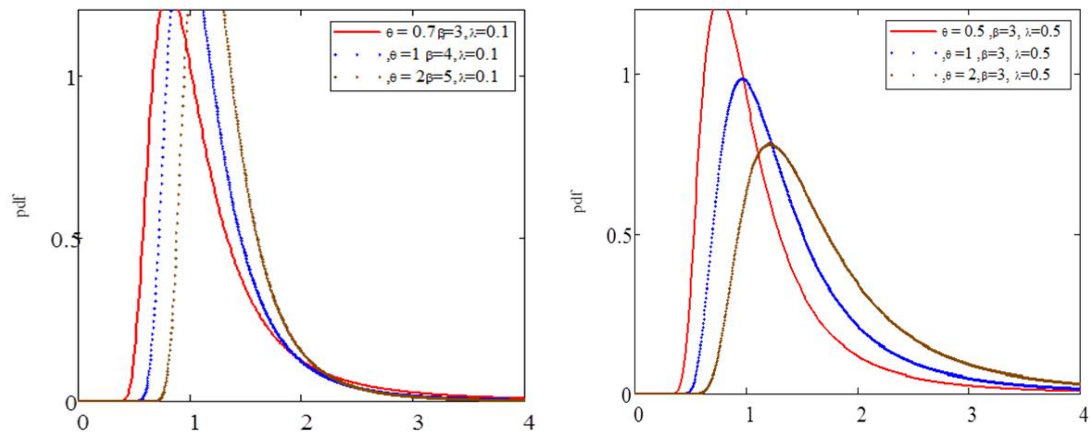

Fig. 5. Plots of the CEIWL densities for some parameter values

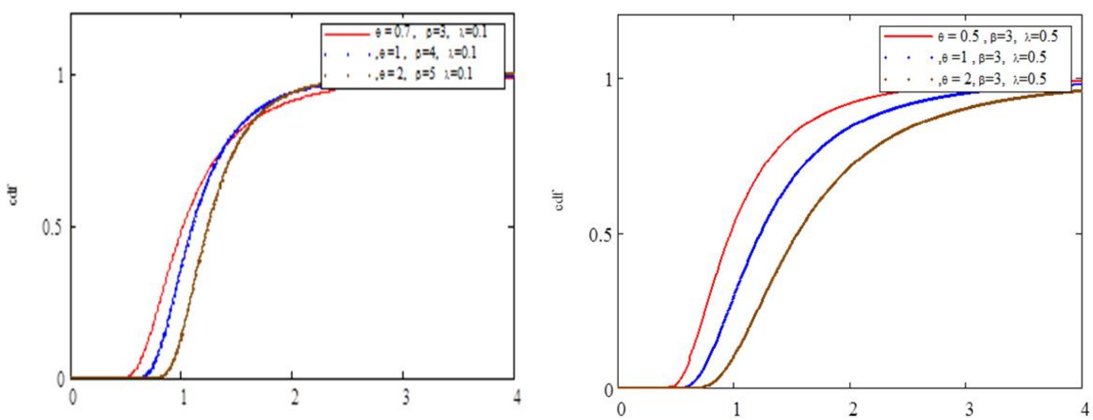

Fig. 6. Plots of the CEIWL distribution function for some parameter values

The quantile function of CEIWL distribution is given by setting $A(\lambda) := -\ln(1-\lambda)$, and $A^{-1}(\lambda) := 1 - e^{-\lambda}$ in (9) , i.e

$$Q(u) = \left[-\frac{1}{\theta}\ln\frac{1-(1-\lambda)u}{\lambda}\right]^{\frac{-1}{\beta}}\ , \tag{27}$$

where u is a uniform random variable on the unit interval (0,1). In particular, the median of the CEIWL distribution, say Q_2, is obtained by setting, $u := \frac{1}{2}$ into (27), i.e.

$$Q_2 = \left[-\frac{1}{\theta}\ln\left(\frac{1+\lambda}{2\lambda}\right)\right]^{\frac{-1}{\beta}},$$

The r-th moment of the CEIWL distribution about the origin is given by substituting $A(\lambda) := -\ln(1-\lambda)$ into (10), i.e.

$$\mu_r^{'} = \sum_{z=1}^{\infty} -\frac{a_z\,\lambda^z}{\ln(1-\lambda)}\,(\theta z)^{\frac{r}{\beta}}\,\Gamma\left(1-\frac{r}{\beta}\right), \qquad r = 1,2,3, \dots \tag{28}$$

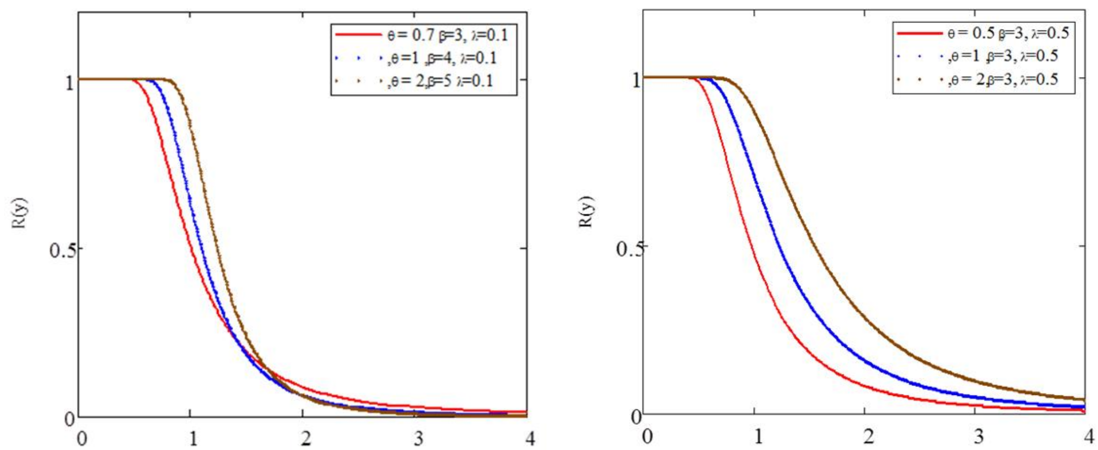

Fig. 7. Plots of the CEIWL reliability functions for some parameter values

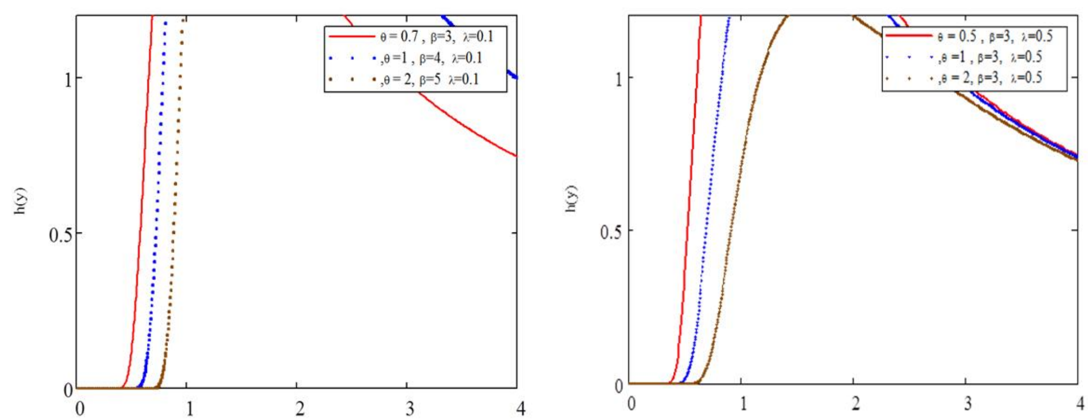

Fig. 8. Plots of the CEIWL hazard rates functions for some parameter values

In particular, setting $r = 1$ in (28), the mean of Y is given by

$$\mathrm{E}(Y) = \sum_{z=1}^{\infty} -\frac{a_z\,\lambda^z}{\ln(1-\lambda)}\,(\theta z)^{\frac{1}{\beta}}\,\Gamma\left(1-\frac{1}{\beta}\right). \tag{29}$$

The variance of the CEIWL distribution takes the following form

$$\mathrm{Var}(Y) = \sum_{z=1}^{\infty} -\frac{a_z\,\lambda^z}{\ln(1-\lambda)}(\theta z)^{\frac{2}{\beta}}\,\Gamma\left(1-\frac{2}{\beta}\right) - \left[\sum_{z=1}^{\infty} -\frac{a_z\,\lambda^z}{\ln(1-\lambda)}(\theta z)^{\frac{1}{\beta}}\,\Gamma\left(1-\frac{1}{\beta}\right)\right]^2.$$

5 Parameter Estimation of the Family

In this section, estimation of the model parameters of CEIWPS family of distributions is obtained by using the maximum likelihood method.

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the CEIWPS family of distributions with parameters θ, β and λ. The likelihood function based on the observed random sample of size n is given by

$$L\left(y;\underline{\psi}\right) = \lambda^n \theta^n \beta^n \big(A(\lambda)\big)^{-n} \prod_{i=1}^{n} y_i^{-(\beta+1)} \prod_{i=1}^{n} e^{-\theta {y_i}^{-\beta}} \prod_{i=1}^{n} A'\left(\lambda\, e^{-\theta {y_i}^{-\beta}}\right).$$

The natural logarithm of the likelihood function, $\ln L \equiv Ln\, L\left(y;\underline{\psi}\right)$, is given by

$$\ln L = n\ln\lambda + n\ln\theta + n\ln\beta - n\ln A(\lambda) - (\beta+1)\sum_{i=1}^{n}\ln y_i - \theta\sum_{i=1}^{n} {y_i}^{-\beta} + \sum_{i=1}^{n}\ln A'\left(\lambda\ e^{-\theta {y_i}^{-\beta}}\right).$$

The maximum likelihood estimators of θ, β, λ, say $\hat{\theta}, \hat{\beta}, \hat{\lambda}$, are obtained by setting the first partial derivatives of ln L to be zero with respect to θ, β and λ as follows,

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} y_i^{-\beta} - \lambda \sum_{i=1}^{n} \frac{A''\left(\lambda\ e^{-\theta {y_i}^{-\beta}}\right)\ e^{-\theta {y_i}^{-\beta}} {y_i}^{-\beta}}{A'\left(\lambda\, e^{-\theta {y_i}^{-\beta}}\right)}, \tag{30}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\ln y_i + \sum_{i=1}^{n} y_i^{-\beta}\ln y_i + \lambda\theta \sum_{i=1}^{n} \frac{A''\left(\lambda\, e^{-\theta {y_i}^{-\beta}}\right) e^{-\theta {y_i}^{-\beta}} y_i^{-\beta} \ln y_i}{A'\left(\lambda\, e^{-\theta {y_i}^{-\beta}}\right)}, \tag{31}$$

and

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \frac{n\,A'(\lambda)}{A(\lambda)} + \sum_{i=1}^{n} \frac{A''\left(\lambda\, e^{-\theta {y_i}^{-\beta}}\right)\ e^{-\theta {y_i}^{-\beta}}}{A'\left(\lambda\, e^{-\theta {y_i}^{-\beta}}\right)}. \tag{32}$$

The maximum likelihood estimator of θ, β and λ, say $\hat{\theta}, \hat{\beta}$ and $\hat{\lambda}$ is obtained by solving numerically the non-linear system of equations $\frac{\partial \ln L}{\partial \theta} \coloneqq 0, \frac{\partial \ln L}{\partial \beta} \coloneqq 0$ and $\frac{\partial \ln L}{\partial \lambda} := 0$.

6 Applications

In this section, the flexibility and potentiality of some special models of EIWPS family are examined using three real data sets. Applications of the CEIWPS distributions based on their sub-models; namely, CEIWP, CEIWL and EIW are considered.

6.1 The First Real Data Set

The first data set is taken from [26], where the vinyl chloride data is obtained from clean upgradient ground –water monitoring wells in mg/L; the data set is as follows:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4,0.2.

The estimator of the unknown parameters of each distribution is obtained by the maximum-likelihood method. In order to compare the three distribution models, various criteria were used. Criteria like; -2logL (-2LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), the correct Akaike information criterion (CAIC) and the Kolmogorov-Smirnov (K-S) statistics are considered for the data set. The "best" distribution corresponds to the smaller values of -2LL, AIC, BIC, CAIC and K-S criteria:

$$\text{AIC} := 2K - 2\text{LL}, \quad CAIC := AIc + \frac{2K(K+1)}{n-K-1}, \quad BIC := k\ln(n) - 2\text{LL},$$

where, k is the number of parameters in statistical model, n is the sample size and L is the maximized value of the log- likelihood function under the considered model.

Also,

$K - S = \sup_y [F_n(y) - F(y)]$, where $F_n(y) := \frac{1}{n}(umber\ of\ observations\ \le y)$, and $F(y)$ denotes the cdf.

Table 2 provides the maximum-likelihood estimates (MLEs), the AIC, BIC, and the CAIC values as well as the Kolmogorov- Smirnov statistics for the first data set.

Table 2. MLEs, AIC values, BIC values, CAIC values and K- S statistics for the first data set

The model	MLEs			Measures				
	θ	β	λ	K-S	-2LL	AIC	CAIC	BIC
CEIWP	0.228	1.084	3.259	0.090	122.801	128.801	129.601	127.35
CEIWL	0.608	0.902	0.181	0.114	195.045	201.045	201.845	199.69
EIW	0.654	0.880		0.113	172.152	176.152	176.539	175.25

Based on the values of AIC, CAIC, BIC and K-S in Table 2, the CEIWP model provide better fit than the CEIWL and EIW models.

Figs. 9 and 10 provide the plots of estimated cumulative and estimated densities of the fitted CEIWP, CEIWL and EIW models for the first data set.

It is clear from Fig. 10 that the fitted density for CEIWP model is closer to the empirical histogram than CEIWL and EIW models.

6.2 The Second Real Data Set

The second data set is obtained from [27]. It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data is as follows:

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

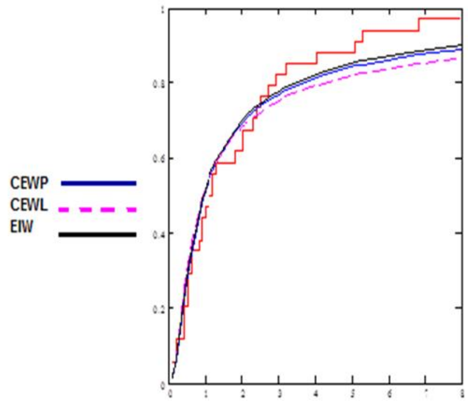

Fig. 9. Estimated cumulative densities for the first data

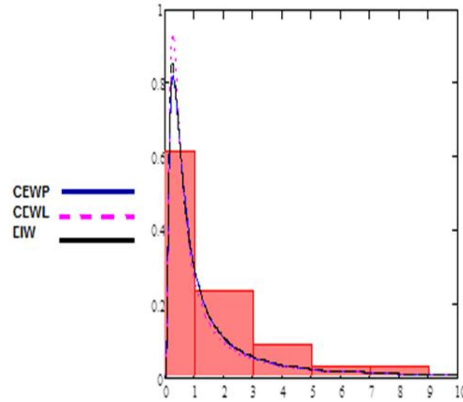

Fig. 10. Estimated densities of models for the first data

The following Table 3 provides the MLEs, the AIC, BIC and CAIC values as well as the K- S statistics for the second data set.

Table 3. MLEs, AIC values, BIC values, CAIC values and K- S statistics for the second data set

The model	MLEs			Measures				
	θ	β	λ	K-S	-2LL	AIC	CAIC	BIC
CEIWP	0.376	1.916	3.474	0.117	85.627	91.627	92.550	90.058
CEIWL	1.019	1.553	0.020	0.153	233.526	239.526	240.326	238.120
EIW	1.025	1.550		0.152	140.186	144.186	144.573	143.249

In order to assess whether the CEIWPS models are appropriate, Fig. 11 provides the histograms of the data set as well as the plots of the fitted CEIWP, CEIWL and EIW density functions. From Fig. 11, it is concluded that the CEIWP distribution is quite suitable for this data set.

Figs. 11 and 12 provide some plots of the estimated cdf as well as the estimated probability densities of the fitted CEIWP, CEIWL and EIW models for the second data set.

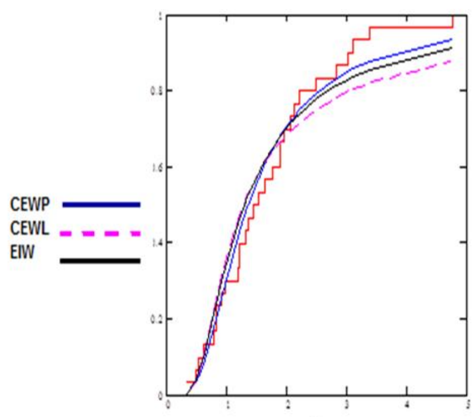

Fig. 11. Estimated cumulative densities for the second data

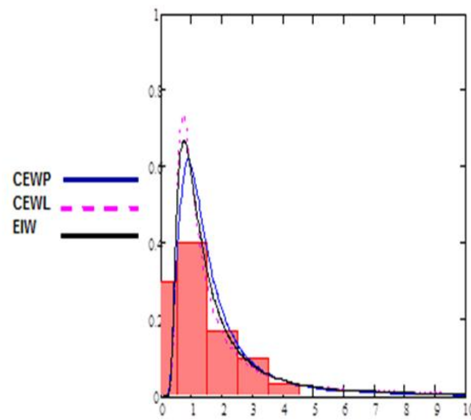

Fig. 12. Estimated densities for the second data

6.3 The Third Real Data Set

The third data set is taken from [28]. The corresponding data are referring to the time between failures for a repairable item. The data set is as follows:

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46,2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30,1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

The following Table 4 provides the MLEs, the AIC, BIC and CAIC values as well as K- S statistics for the third data set.

Table 4. MLEs, AIC values, BIC values, CAIC values and K- S statistics for the third data

The model	MLEs			Measures				
	θ	β	λ	K-S	-2LL	AIC	CAIC	BIC
CEIWP	0.680	1.104	0.279	0.1334	101.368	107.368	108.292	105.80
CEIWL	0.747	1.075	0.017	0.1339	221.411	227.411	228.334	225.82
EIW	0.752	1.073		0.1337	127.816	131.816	132.260	130.70

The values in Table 4, indicate that the complementary exponentiated inverted Weibull Poisson model is a strong competitor among other distributions that was used here to fit this data set. Figs. 13 and 14 provide some plots of the estimated cdf and pdf of the fitted CEIWP, CEIWL and EIW models for the third data set.

In Fig. 14 the fitted densities of the CEIWP, CEIWL and EIW models are compared to the empirical histogram of the observed third data set. Based on the plots of Fig. 14, one can notice that the fitted density of the CEIWP model is closer to the empirical histogram than the corresponding densities of the CEIWL and EIW models.

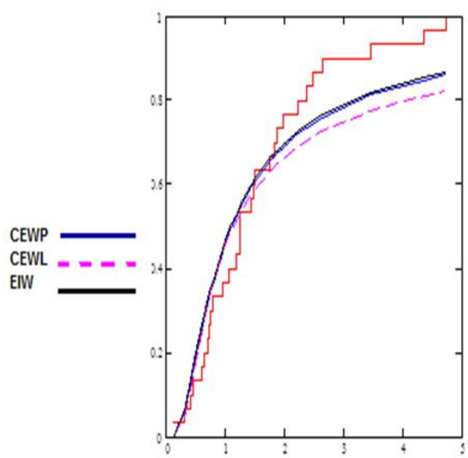

Fig. 13. Estimated cumulative densities for the third data

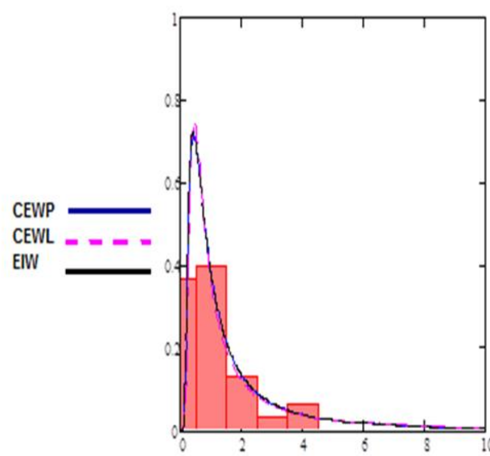

Fig. 14. Estimated densities of the third data

7 Conclusions

In this paper, a new family of lifetime distributions, called the complementary exponentiated inverted Weibull power series was introduced. This family was obtained by compounding the exponentiated inverted

Weibull and power series distributions. The properties of the proposed family were discussed, including the quantile function, the moments and the moment generating function. Some distributions of order statistics were also obtained. The estimation of the model parameters was performed by the maximum likelihood method. Two special sub-models of the new family were investigated, namely, the complementary exponentiated inverted Weibull Poisson distribution and the complementary exponentiated inverted Weibull logarithmic distribution. Some mathematical properties of the new distributions were also discussed. Finally, the complementary exponentiated inverted Weibull Power series models were fitted to real data sets revealing the flexibility and potentiality of the introduced CEIWPS family of distributions.

Acknowledgement

The Authors would like to express their gratitude to the Editor and the reviewers for useful remarks, careful reading and for providing very constructive comments that led to improvement of presentation of the current article.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Adamidis K, Loukas S. A life time distribution with decreasing failure rate. Statistics and Probability Letters. 1998;39:35-42.

[2] Rezaei S, Tahmasbi R. A new lifetime distribution with increasing failure rate: Exponential truncated poisson. Journal of Basic and Applied Scientific Research. 2012;2(2):1749-1762.

[3] Mahmoudi E, Shiran M. Exponentiated Weibull-geometric distribution and its applications; 2012. Available:http://arxiv.org/abs/1206.4008

[4] Louzada F, Roman M, Cancho VG. The complementary exponential geometric distribution: Model, properties, and a comparison with its counterpart. Computational Statistics and Data Analysis. 2011; 55(8):2516-2524.

[5] Mahmoudi E, Sepahdar A. Exponentiated Weibull Poisson model, properties and applications. Journal of Mathematics and Computers in Simulation. 2012;92:76-97.

[6] Leahu A, Munteanu BG, Cataranciuc S. On the lifetime as the maximum or minimum of the sample with power series distributed size. ROMAI Journal. 2013;9:119-128.

[7] Yamachi CY, Roman M, Louzada F, Franco MAP, Chanco VG. The exponentiated complementary exponential geometric distribution. Journal of Modern Mathematics Frontier Statistics. 2013;2(2): 78-83.

[8] Louzada F, Marchi V, Carpenter J. The complementary exponentiated exponential geometric lifetime distribution. Journal of Probability and Statistics; 2013.
Available:http://dx.doi.org/10.1155/2013/502159

[9] Louzada F, Yamachi CY, Marchi VAA, Franco MAP. The long-term exponentiated complementary exponential geometric distribution under a latent complementary gausses framework. Tendˆencias em Matem´ atica Aplicada e Computacional. 2014;15(1):19-35.
Available:http://www.scielo.br/pdf/tema/v15n1/a03v15n1.pdf

[10] Al-Zahrani B, Sagor H. The Poisson-Lomax distribution. Revista Colombiana de Estadistica. 2014; 37(1):223-243.

[11] Chowdhury S. Compound generalized Weibull distributions-a unified approach. Indian Institute of Management Kozhikode. 2014;6:1-18.

[12] Gui W, Zhang S, Lu X. The Lindley-Poisson distribution in lifetime analysis and its properties. Hacttepe Journal of Mathematics and Statistics. 2014;43(6):1063-1077.

[13] Pararai M, Liyanage GW, Oluyde BO. A New class of generalized power Lindely distribution with applications to lifetime data. Theoretical Mathematics and Applications. 2015;5(1):53-96.

[14] Hassan AS, Assar SM, Ali, KA. The complementary Poisson-Lindley class of distributions. International Journal of Advanced Statistics and Probability. 2015;3(2):146-160.

[15] Morais AL, Barreto-Souza W. A compound class of Weibull and power series distributions. Computational Statistics and Data Analysis. 2011;55(3):1410-1425.

[16] Mahmoudi E, Jafari AA. Generalized exponential Power series distributions. Computational Statistics and Data Analysis. 2012;56:4047-4066.

[17] Silva RB, Bourguignon M, Dais CRB, Cordeiro GM. The compound class of extended Weibull power series distributions. Computational Statistics and Data Analysis. 2013;58:352-367.

[18] Leahu A, Munteanu BG, Cataranciuc S. Max-Erlang and min- Erlang power series distributions as two new families of lifetime distribution. Buletinul Academiei de Stiinte A Republicii Moldova. Mathematica. 2014;75:60-73.

[19] Silva RB, Cordeiro GM. The Burr XII power series distributions: A new compounding family. Brazilian Journal of Probability and Statistics. 2015;29(3):565-589.

[20] Liyanage GW, Pararai M. The Lindley power series class of distributions: Model, properties, and applications. Journal of Computations and Modeling. 2015;5(3):35-80.

[21] Shafiei S, Darijani S, Saboori H. Inverse Weibull power series distributions: Properties and applications. Journal of Statistical Computation and Simulation; 2015.
Available:http://www.tandfonline.com /loi/gscs20

[22] Keller AZ, Kamath ARR, Perera UD. Reliability analysis of CNC machine tools. Reliability Engineering. 1982;3(6):449–473.

[23] Flaih A, Elsalloukh H, Mendi E, Milanova M. The exponentiated inverted Weibull distribution. Applied Mathematics and Information Sciences. 2012;2:167-171.

[24] David HA. Order statistics. Second edition. Wiley. New York; 1981.

[25] Gradshteyn IS, Ryzhik IM. Table of integrals, series and products. San Diego: Academic Press; 2000.

[26] Bhaumik DK, Kapur KG, Gibbons RD. Testing parameter of a gamma distribution for small samples. Technimetrics. 2009;51(3):326-334.

[27] Hinkley D. On quick choice of power transformations. Journal of the Royal Statistical Society, Series (c), Applied Statistics. 1977;26(1):67–69.

[28] Murthy DNP, Xie M, Jiang R. Weibull models, series in probability and statistics. John Wiley, New Jersey; 2004.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/12518